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Dynamic optimization of branching diffusion processes

Stochastic Control's lens on particle systems and their scaling limits

Thesis supervised by Idris Kharroubi

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Être comme tout le monde était la visée générale, l'idéal à atteindre. L'originalité passait pour de l'excentricité, voire le signe qu'on en a un grain.

La Honte - Annie Ernaux

The skill I was learning was a crucial one, the patience to read things I could not yet understand.

Educated: A Memoir - Tara Westover

Oh, felice mio core!
Dopo tanti tormenti
pur giungesti alla sfera dei contenti.

Ariodante - George Frideric Handel

Dynamic optimization of branching diffusion processes
Stochastic Control's lens on particle systems and their scaling limits

Abstract

The goal of this thesis is to uncover interesting structures occurring in the intersection of three distinct fields: stochastic control theory, branching diffusion processes, and McKean–Vlasov dynamics. In the initial phase, we investigate potential extensions of the stochastic target problem and the optimal stopping problem within the context of branching processes. By constraining our examination to cost functions that respect the inherent symmetry of the problem, we show how the optimization of a global criterion can be recast as finite-dimensional optimization challenges through the utilization of a branching property. This finding paves the way to a differential characterization. Using a dynamic programming approach, we prove the value function is the unique viscosity solution to an HJB equation.
The second part of this work delves into the theory of controlled branching diffusion processes, under a symmetrical structure in the cost function with respect to particle labeling. Exploring a relaxed formulation, we rewrite the control problem as the minimization of a lower semicontinuous function within a compact domain. This formulation, therefore, provides theoretical guarantees regarding the existence of a globally optimal solution. This abstract setting paves the way to scaling limits for these processes, leading to the class of controlled superprocesses. Within this dynamical framework, we establish an HJB equation in the space of finite measures. Moreover, for specific cost functions, we go back to the initial approach, retrieving regular solutions for the control problem through a branching property and finite-dimensional optimization.

Keywords: stochastic control, stochastic target control, optimal stopping, relaxed control, branching diffusion process, superprocesses, dynamic programming principle, Hamilton–Jacobi–Bellman equation, viscosity solution, martingale representation

Résumé

Cette thèse se trouve à l'intersection de trois sujets différents : la théorie du contrôle stochastique, les processus de diffusion branchants et la dynamique de McKean–Vlasov. Initialement, nous étudions les extensions du problème de la cible stochastique et du problème de l'arrêt optimal pour des processus de branchement. Pour des fonctions de coût qui respectent la symétrie inhérente au problème, nous montrons comment l'optimisation d'un critère global peut être transformée en un problème à dimension finie grâce à l'utilisation d'une propriété de branchement. Cette constatation ouvre la voie à une caractérisation différentielle. En utilisant une approche de programmation dynamique, nous prouvons que la fonction de valeur est l'unique solution de viscosité d'une équation de HJB.
La deuxième partie de ce travail approfondit la théorie des processus branchants contrôlés, sous une structure symétrique de la fonction de coût par rapport à l'étiquette des particules. En explorant une formulation relâchée, nous réécrivons le problème de contrôle comme la minimisation d'une fonction semi-continue inférieurement à l'intérieur d'un compact. Ce point de vue fournit donc des garanties théoriques quant à l'existence d'une solution optimale. Ce cadre abstrait ouvre la voie à des limites d'échelle pour ces processus, conduisant à la classe des superprocessus contrôlés. Nous établissons ainsi une équation de HJB dans l'espace des mesures finies. De plus, pour des fonctions de coût de type exponentiel, nous revenons à l'approche initiale, retrouvant des solutions régulières pour le problème de contrôle grâce à une propriété de branchement et à une optimisation en dimension finie.

Mots clés : contrôle stochastique, cible stochastique, arrêt optimal, contrôle relaxé, processus de branchement de diffusion, superprocessus, principe de programmation dynamique, équation de Hamilton–Jacobi–Bellman, solution de viscosité, représentation martingale

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Contents

Preface

> **Who controls the past controls the future. Who controls the present controls the past.**
>
> George Orwell - *1984*

George Orwell, in *1984*, describes a world in which people are manipulated by the government. With this sentence, he attempts to decrypt how this could be achieved. This concept hides a deeper reflection: when do the goals of the individual coincide with the goals of society? From a mathematical point of view, this comes down to trying to work out when a certain utility function on the dynamics of the whole population can be broken down into its components and recast into an optimization problem of the behavior of a single representative individual. To represent the problem in this way would necessarily reconcile the macroscopic and the microscopic scales.

In this thesis, we investigate this link between macroscopic and microscopic optimization by developing a theory of controlled populations. The above quote allows us to reflect on multiple aspects: How does the past, and in particular the choices made in the past, influence the future? How can the dependency between control and global population dynamics be modeled? What are the ways to evaluate the decisions of the individual from a global point of view? How can this be done such that the maximization of a global reward boils down to a maximization problem from the individual's perspective?

In crafting a comprehensive theory of controlled population, *1984* lays the foundation for our scientific inquiry, though it does not explicitly reveal its core elements. Within its pages, we identify a fundamental premise for our scientific pursuit. We intend to revisit and adapt these concepts, aligning them with mathematical literature to systematically formulate the inquiries we aim to tackle.

> **Perhaps one did not want to be loved so much as to be understood.**
>
> George Orwell - *1984*

A tool that, once it is *understood*, shows hidden links between different fields of mathematics is the *Feynman–Kac formula.* This result is a fundamental and influential achievement in mathematical analysis and stochastic calculus; it plays a crucial role in establishing a profound connection between partial differential equations (PDEs) and stochastic processes. This formula provides a mechanism to solve specific types of PDEs, by relating them to expectations of functionals of stochastic processes. This remarkable link has far-reaching applications spanning various fields, including physics, finance, and engineering.

Connecting the deterministic dynamics described by PDEs with random processes offers a versatile approach to handling scenarios where the evolution of a system exhibits both deterministic and stochastic elements. For instance, in financial mathematics, the formula has been instrumental in the study of option pricing: by incorporating a stochastic process to model the underlying asset price, the Feynman–Kac formula allows for the evaluation of option prices in a probabilistic framework.

It gives deep insights into the connections between probability theory, stochastic calculus, and PDEs. Such a formula allows us to calculate approximations of solutions of second-order linear PDEs by Monte Carlo methods, approximating trajectories of SDEs, thus not suffering from the curse of dimensionality. Moreover, considering the infinitesimal generator for stochastic processes gives a new perspective to the understanding of these dynamics. This makes it possible to consider complex dynamics while still having analytical tools to study them.

Understanding what connections the Feynman–Kac formula conceals has been central to research in probability theory. Many have focused on how to extend it, as we can read in [131]:

> *There has been in the past at least three ways of extending the Feynman-Kac formula to nonlinear equations. One is to replace the diffusion $\{X_t\}$ by a controlled diffusion (see Fleming, Soner [81]), the second is to replace it by a branching-diffusion process (or a superprocess, see e.g. Dynkin [59]), the third is to replace it by a nonlinear Markov process in the sense that the evolution of X_t depends not only of X_t but also on its probability law, see e.g. McKean [119].*

These three axes of research constitute major mathematical fields to which research has been extensively devoted. By combining them, we will try to answer our initial question of how to model controlled populations. We outline now the key ideas that will guide our research for this purpose.

> **The consequences of every act are included in the act itself.**
>
> George Orwell - *1984*

Controlling means modeling the consequences of making a certain decision. These effects can be of two types: one explicit and one implicit. The first seeks the optimization of a given utility function. The second, as noted by George Orwell, tells us that the very fact of making a decision is a decision in itself. This second characteristic entails a modification of the dynamics we are observing.

These stylized facts are the cornerstones of *stochastic control theory.* This branch of mathematics deals with optimal decision-making, where both random elements and control actions are present. It provides a framework for studying and solving problems where the objective is to find the best control strategy to optimize a certain criterion under uncertainty.

A fundamental concept in this field is the Dynamic Programming Principle (DPP). The DPP states that an optimal control strategy can be obtained by breaking down the global problem into smaller ones and solving them recursively. This enforces that adopting optimal behavior at a given time is consistent with our past and future counterparts. Therefore, trusting our future decisions, we solve the problem by decomposing it into more manageable sub-problems, deriving optimal control strategies iteratively. This principle offers valuable insights into optimal control policies and is the building block for efficient algorithms to find approximations of the optimal solutions.

For time-continuous dynamics, the proof of this result may be quite technical and is generally based on the *martingale problem.* The martingale problem is a reformulation of the strong stochastic control problem into its weak version. It involves specifying a family of probability measures on a given state space. This formulation defines a collection of paths that the stochastic process can evolve, encompassing both random evolution and control actions. By defining the problem in terms of a martingale problem, the dynamic programming principle provides a systematic framework for solving stochastic control problems.

The DPP lays the groundwork for the derivation of the dynamic programming equation, also known as the *Hamilton–Jacobi–Bellman (HJB) equation*. This is a nonlinear PDE that characterizes the value function, relating it to the state variables, the control variables, and the underlying system dynamics. It serves as a necessary condition for the optimality of the control strategy.

Solving the HJB equation provides a means to determine optimal control strategies that maximize the value function. It gives a complete characterization of the optimal policy in terms of state variables and system dynamics. In this way, in conjunction with the HJB equation, the DPP closes the circle by offering a systematic computational approach to identifying the optimum in sequential decision problems.

> **The family could not actually be abolished, and, indeed, people were encouraged to be fond of their children, in almost the old-fashioned way.**
>
> George Orwell - *1984*

Modeling population dynamics involves working with concepts such as generations, genealogy, and random reproduction of members of this population. One way to describe mathematically these objects is through *branching diffusion processes*. This area studies stochastic processes that exhibit both branching and diffusive behavior. The latter means that the feature we are interested in evolves as a continuous-time diffusion, inheriting the characteristics of the previous generation at each branching event.

This class of processes can be described in two different ways. The first is to define them as dynamics evolving in time and living in the space of finite measures. In this description, we focus on the distribution of individuals or particles at a given time. As time progresses, the process incorporates both a branching mechanism, in which individuals generate offspring, and a diffusive behavior, which accounts for the random movement or spreading of individuals. This point of view treats the population as a whole, presenting the characteristics of the individual as homogenized in a large global process.

Alternatively, the second formulation defines this class of dynamics as real-valued processes indexed by a Galton–Watson tree. This tree structure represents the genealogical relationships among individuals in a branching process. The process is seen as a collection of individual lineages, corresponding to a distinct path in the Galton–Watson tree, along which diffusive dynamics are attached. When branching events occur, the endpoint of the branching particle initiates the dynamics for its offspring, with the index determined by the hierarchical structure of the tree. This modeling highlights the underlying skeleton of the generated genealogy and it is closer to a description of the individual's behavior while making it more complex to directly access the population seen as a whole.

These two descriptions provide complementary insights into the behavior and properties of these processes. Indeed, the perspective of evolving in the space of finite measures emphasizes the overall distribution and spread of individuals in the population, while the Galton–Watson tree representation focuses on genealogical relationships and individual trajectories. This is the reason why they are employed to model different phenomena.

> **It is not possible for any thinking person to live in such a society as our own without wanting to change it.**
>
> George Orwell - *1984*

As part of a population, we both influence and are influenced by our surroundings. This notion is the foundation of *McKean–Vlasov dynamics*, where the interplay between microscopic and macroscopic behaviors is explored. This field studies stochastic processes where the actions of each individual are influenced by the average or mean field generated by the entire population. This framework takes into account interactions and feedback effects among individuals, giving rise to intricate dynamics that exhibit a combination of individual randomness and collective behavior.

A central result in these dynamics is the so-called *propagation of chaos*. This phenomenon refers to the convergence as the number of individuals tends to infinity of the empirical mean associated with individual processes to the mean field measure. This convergence implies that the collective behavior of the population can be accurately described by a deterministic mean field equation, neglecting stochastic fluctuations at the individual level. This generalization of the law of large numbers provides a bridge between microscopic and macroscopic levels of analysis for this kind of dynamics.

Remarkable generalizations of these concepts and methodologies are the theory of Mean Field Games (MFG) and Mean Field Control (MFC). MFG extends the framework of McKean–Vlasov dynamics to include strategic interactions among individuals, where each player's decision-making takes into account the average behavior of the entire population. MFC deals with optimal control problems in systems where the control actions of each agent depend on the average behavior of the entire population. These two generalizations find multiple applications in economics, finance, social sciences, robotics, energy management, and network optimization. They provide insights into phenomena such as crowd behavior, traffic flow, and resource allocation.

Overall, McKean–Vlasov dynamics offer a mathematical framework to analyze and understand complex systems where individual stochastic behavior is influenced by collective interactions. In this context, the branching of particles is not taken into account. When this happens, a different zoology of problems opens up, where the frequency of reproduction and temporal dimension are closely intertwined.

Introduction

Outline of the current chapter

Mathematical context

We will now shift our focus to delve deeper into the key tools that will be extensively employed in the following chapters. The tools in these disciplines provide the foundational knowledge and methodologies that contribute to the comprehensive exploration and analysis conducted in this thesis. By examining these major areas, we can gain a deeper understanding of the adopted multifaceted approach.

Fix two functions $(b, \sigma) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ that are Lipschitz continuous in the spatial variable $x \in \mathbb{R}^d$, uniformly in the temporal variable $t \in [0,T]$, for a finite horizon $T > 0$. For a function $f \in C_b^2(\mathbb{R}^d)$, bounded with bounded second order derivatives, define the function $Lf : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ as

$$Lf(t,x) := b(t,x)^\top Df(x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(t,x)D^2 f(x)\right), \tag{0.0.1}$$

where Df (resp. $D^2 f$) denotes the first order (resp. second order) derivative of the function f. It is known that there exists a unique strong solution $\{X_s^{t,x}\}_{s\geq t}$ of the following Stochastic Differential Equation (SDE)

$$X_s^{t,x} = x + \int_t^s b\left(u, X_u^{t,x}\right) du \int_t^s \sigma\left(u, X_u^{t,x}\right) dB_u, \tag{0.0.2}$$

where $\{B_u\}_{u\geq 0}$ is a standard Brownian motion (see, *e.g.*, [105, Theorem 2.5.7]).

The following result is the so-called Feynman–Kac formula, as presented in [151, Proposition 2.6].

Proposition 0.0.1. *Fix a function $g : \mathbb{R}^d \to \mathbb{R}$. Assume that the function*

$$[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto v(t,x) := \mathbb{E}\left[g(X_T^{t;x})\right] \tag{0.0.3}$$

is $C^{1,2}([0,T] \times \mathbb{R}^d)$. Then, the function v solves the following PDE

$$\begin{cases} \partial_t v + Lv = 0, \\ v(T,\cdot) = g. \end{cases} \tag{0.0.4}$$

Equation (0.0.3) indicates a direct approach for approximating the solution of (0.0.4) using Monte Carlo theory. By employing methods like Euler schemes to approximate (0.0.2), the curse of dimensionality is no longer a limiting factor. Conversely, the theory centered on the infinitesimal generator L enables the consideration of more intricate dynamics than those described by (0.0.2).

Area 1: Stochastic control

Stochastic control theory emerged in the 1960s and has, since then, experienced significant development and diversification, driven by the need to model and address various applications. Several monographs can be found on this topic, for example, [81, 106, 126, 138, 151, 156]. We base this brief introduction mainly on [138, Chapter 3] and [151, Chapter 3]. We present the fundamental concepts and definitions that underpin stochastic control without going into formal proofs, which can be easily found in the previous references.

Standard controlled diffusion processes

Stochastic control provides a powerful framework for decision-making in the presence of uncertainty. It involves optimizing an objective function by manipulating control variables in response to the evolution of stochastic processes.

We now give a brief presentation of the probabilistic setting. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual conditions, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration representing the information available up to time t. We consider a controlled dynamic system defined by the state process $X = \{X_t\}_{t\geq 0}$, a controlled process $\alpha = \{\alpha_t\}_{t\geq 0}$, and a standard Brownian motion $B = \{B_t\}_{t\geq 0}$, which captures the randomness in the environment. Here, X_t represents the state of the system at time t, α_t denotes the control at time t, and B_t represents the exogenous randomness.

The control process $\alpha = \{\alpha_t\}_{t\geq 0}$ is progressively measurable (with respect to $\mathbb{F}$) and takes values in a given control set A, subset of $\mathbb{R}^m$, for $m \geq 1$. The evolution of the state process X is described by the following SDE

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t, \tag{0.0.5}$$

where $b : \mathbb{R}^d \times A \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times A \to \mathbb{R}^{n\times d}$ are deterministic continuous functions representing the drift and diffusion coefficients, respectively. We assume that there exists a constant $L > 0$ such that

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq L|x-y|, \tag{0.0.6}$$

$$|b(x,a)| + |\sigma(x,a)| \leq L(1 + |x| + |a|), \tag{0.0.7}$$

for any $x, y \in \mathbb{R}^d$, $a \in A$. We denote $\mathcal{A}$ as the set of progressively measurable controls α such that

$$\mathbb{E}\left[\int_0^T |\alpha_t|^2 dt\right] < \infty, \tag{0.0.8}$$

and we say that $\mathcal{A}$ is the set of admissible controls.

Fix a given time horizon $T > 0$. Consider $f : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}$ (resp. $g : \mathbb{R}^d \to \mathbb{R}$) the running (resp. terminal) cost function. Suppose f and g continuous with quadratic growth in x and linear in a, *i.e.*, there exists a constant $C > 0$ such that

$$|f(x,a)| + |g(x)| \leq C(1 + |x|^2 + |a|) \tag{0.0.9}$$

for some constant $C > 0$ and any $(x,a) \in \mathbb{R}^d \times A$. For $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, define the following associated cost

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T f(X_s^{t,x}, \alpha_s) ds + g(X_T^{t,x})\right],$$

where $\{X_s^{t,x}\}_{s \geq t}$ is the unique strong solution of the SDE (0.0.5) starting at x at time t (see, *e.g.*, [151, Theorem 3.1]).

The objective of the controller is to choose an optimal control strategy that minimizes a performance criterion associated with f, g. This means studying the following optimization problem

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} J(t,x,\alpha).$$

Relaxed formulation

The relaxed formulation of the control problem was first introduced in [17, 79, 80] and generalized in [64] to include controls in the diffusion term. Since then, it has been widely adopted in the field, as evidenced by its usage in various studies (see, *e.g.*, [8, 27, 50, 86]). The essence of the relaxed formulation lies in the consideration of a space known as the *space of generalized actions*. This space is the subset of Radon measures on $\mathbb{R}_+ \times A$, with the property that their projection onto $\mathbb{R}_+$ corresponds to the Lebesgue measure.

By employing this generalization, we can transform the control problem into a related martingale problem. The martingale problem is a fundamental concept closely associated with stochastic control. It serves as a means to characterize the existence and uniqueness of a solution to SDEs, as presented in [67]. In the context of relaxed control problems, the martingale problem offers a rigorous mathematical framework for examining the optimality and feasibility of control strategies.

The martingale problem involves specifying a family of probability measures $\{\mathbb{P}_x\}_{x \in \mathbb{R}^d}$ indexed by the initial state x. These measures are defined on the space of paths and satisfy certain consistency conditions. The solution to the martingale problem corresponds to the existence of a process $\{X_t\}_{t \geq 0}$ satisfying the given SDE and the associated filtration.

This mathematical object reveals profound connections with the control problem on two fronts. Firstly, if a specific class of stochastic controlled processes can be reformulated as a martingale problem, then the tools and techniques of control theory can be effectively applied. Secondly, by employing a martingale problem framework, the focus shifts towards probabilities associated with the space of trajectories. This shift enables us to work within a space where the topology is more easily described, as compared to the space of controls, where understanding the convergence of controls and its impact on the controlled processes is inherently more complex. This reformulation ultimately facilitates the rigorous definition of a dynamic programming principle, as described in [45, 69, 70].

Dynamic Programming Principle

The *Dynamic Programming Principle* (DPP) serves as a cornerstone of stochastic control, providing a recursive relationship that links the value function v at different time points. This principle enables the decision-making process to be carried out incrementally, allowing for optimal trajectory selection over small time intervals within the control problem. We present a version of this result, as presented in [138, Theorem 3.3.1].

Theorem 0.0.1. *Assume that v is continuous and fix $(t, x) \in [0, T] \times \mathbb{R}^d$. Let θ be a stopping time with values in $[t, T]$. Then, we have that*

$$v(t,x) = \inf_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\int_t^\theta f(X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x}) \right]$$

where $\mathcal{A}_t$ is the set controls living in $\mathcal{A}$ that are independent of $\mathcal{F}_t$.

Although the DPP offers valuable insights into optimizing stochastic control problems, it introduces significant challenges regarding measurability, particularly when concatenating controls. Ensuring the measurability of the resulting control process presents a substantial obstacle in the analysis. In this regard, the martingale problem formulation plays a crucial role. Moreover, this reformulation provides a robust framework for establishing a rigorous proof of the DPP (see, *e.g.*, [45, 69, 70]).

HJB equation

The DPP paves the way for the formulation of the associated Hamilton–Jacobi–Bellman (HJB) equation. The HJB equation is a PDE that characterizes the value function in terms of the system's dynamics and a performance criterion. By solving the HJB equation, one can obtain valuable insights into the optimal control strategies and associated value functions in stochastic control problems.

Under certain assumptions over the regularity of the value function, from Itô's formula, we obtain the verification theorem for the given control problem. This result serves as a powerful tool to assess the optimality of a candidate control strategy by comparing it to the HJB equation.

Consider mow the linear second order operator L^a associated to the controlled process $\{X_t\}_t$ controlled by the constant control process $a \in A$

$$L^a\varphi(x) = b(t,x)^\top D\varphi(x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x,a)D^2\varphi(x)),$$

for $\varphi \in C^2(\mathbb{R}^d)$, where D and D^2 denote the gradient and the Hessian operators. With these elements, we can now give the verification theorem, as given in [138, Theorem 3.5.2].

Proposition 0.0.2. *Let w be a function in $C^{1,2}([0,T) \times \mathbb{R}^d) \cap C^0([0,T] \times \mathbb{R}^d)$ satisfying a quadratic growth condition, i.e., there exists a constant $C > 0$ such that*

$$|w(t,x)| \leq C(1 + |x|^2),$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$
(i) Suppose that

$$-\frac{\partial}{\partial t} w(t, x) - \inf_{a \in A} \{L^a w(t, x) + f(x, a)\} \leq 0, \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^d,$$
$$w(T, x) \leq g(x), \quad \text{for } x \in \mathbb{R}^d.$$

Then, $w \leq v$ *on* $[0, T] \times \mathbb{R}^d$.
(i) Suppose further that $w(T, x) = g(x)$*, and there exists a measurable function* $\hat{\alpha}(t, x)$*, for* $(t, x) \in [0, T) \times \mathbb{R}^d$*, valued in* A *such that*

$$\begin{aligned} -\frac{\partial}{\partial t} w(t, x) - \inf_{a \in A} \{L^a w(t, x) + f(x, a)\} &= -\frac{\partial}{\partial t} w(t, x) - L^{\hat{\alpha}(t,x)} w(t, x) - f(x, \hat{\alpha}(t, x)) \\ &= 0, \end{aligned}$$

the SDE

$$dX_t = b(X_t, \hat{\alpha}(t, X_t))dt + \sigma(X_t, \hat{\alpha}(t, X_t))dB_t,$$

admits a unique solution, denoted by $\{\hat{X}_s^{t,x}\}_s$*, given an initial condition* $X_t = x$*, and the process* $\{\hat{\alpha}(s, \hat{X}_s^{t,x})\}_{s \in [t,T]}$ *lies in* $\mathcal{A}_t$*. Then,*

$$w = v \quad \text{on } [0, T] \times \mathbb{R}^d,$$

and $\hat{\alpha}$ *is an optimal Markovian control.*

Upon comparing this outcome with Proposition 0.0.1, we observe a clear connection between this class of random evolution and the theory of PDE. Nevertheless, establishing the a priori regularity of the value function itself poses significant challenges. This difficulty arises due to the nonlinearity of the involved PDE. The smoothness characteristics of the cost functions, f and g, directly impact the existence of smooth solutions to the HJB equation.

This is where the theory of viscosity solutions comes into play since a more robust and flexible framework is required to address these challenges. Viscosity solutions provide a powerful approach to dealing with nonlinear PDEs without relying on strict smoothness assumptions. They allow for the study of solutions that exhibit discontinuities or lack classical smoothness, providing a more general and comprehensive understanding of the regularity properties of the value function in stochastic control problems. We refer to [12, 13, 138, 151] to know more on the subject.

Stochastic target problem

Two prominent examples that exemplify how flexible stochastic control tools are can be found in the *stochastic target problem* and the *optimal stopping problem*. These problems serve as key illustrations of the versatility and effectiveness of the techniques so far introduced.

Stochastic target problems involve determining an optimal control strategy to achieve a specific target in the presence of uncertainty. This theory finds various application in finance (see, *e.g.*, [23, 26, 34, 147]) In a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0})$ satisfying the usual conditions, consider the same setting of the stochastic control problem. The state process is defined as follows: given the initial data $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, and $\alpha \in \mathcal{A}$, let the controlled process $(X^{t,x,\alpha}, Y^{t,x,y,\alpha})$ be the solution of the stochastic differential equation

$$\begin{aligned} dX_t &= b(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dB_t, \\ dY_t &= b_Y(X_t,Y_t,\alpha_t)dt + \sigma_Y(X_t,Y_t,\alpha_t)dB_t, \end{aligned}$$

with initial data (t,x,y).

As before, the functions $(b,\sigma) : \mathbb{R}^d \times A \to \mathbb{R}^d \times \mathbb{R}^{n\times d}$ and $(b_Y,\sigma_Y) : \mathbb{R}^d \times \mathbb{R} \times A \to \mathbb{R} \times \mathbb{R}^{n\times 1}$ are continuous and satisfy (0.0.6)-(0.0.7). We denote $\mathcal{A}$ as the set of progressively measurable controls α satisfying (0.0.8) and we say that $\mathcal{A}$ is the set of admissible control. Fix a given time horizon $T > 0$ and a continuous function $g : \mathbb{R}^d \to \mathbb{R}$ satisfying (0.0.9). For $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, we define the stochastic target problem by

$$v(t,x) = \inf \left\{ y \in \mathbb{R} : Y^{t,x,y,\alpha} \geq g(X^{t,x,\alpha}), \mathbb{P} - \text{a.s. for some } \alpha \in \mathcal{A} \right\}.$$

The stochastic target problem holds a direct connection to the concept of super-replication cost in finance. Super-replication is a hedging strategy used to replicate a given financial derivative using a smaller set of liquid traded assets. The super-replication cost represents the discrepancy between the actual price of the derivative and the price obtained through the super-replication strategy.

Typically, the resolution of the stochastic target problem involves applying a (geometric) DPP technique and establishing a characterization through an HJB equation. This approach introduces a constraint on the gradient of the value function, leading to a lack of smoothness. Consequently, obtaining precise analytical solutions is often challenging. Thus, the adoption of the viscosity solution for the associated HJB equation becomes essential in addressing this complexity. We refer to [145, 146] for a more detailed description.

Optimal stopping

The optimal stopping focuses on determining the optimal time to take a particular action to maximize a given objective. We reefer to [63, 103, 104] for more detailed description of this framework.

In this setting, the state process X satisfies that SDE (0.0.2), with initial data (t,x). In the financial literature, this process aims at representing the underlying state variable or asset price. We denote $\mathcal{T}_t$ as the set of stopping times taking values in $[t,T]$.

The objective of the optimal stopping problem is to determine the optimal stopping time that minimizes a specific objective function J as follows

$$J(t,x,\theta) = \mathbb{E}\left[\int_0^\theta f(X_s^{t,x},\alpha_s)ds + g(X_\theta^{t,x}) \right],$$

with $\theta \in \mathcal{T}_t$. The stopping time θ represents the time at which a decision is made to stop the process and take a particular action. Therefore, we aim to study the following optimization problem

$$v(t,x) = \inf_{\theta\in\mathcal{T}_t} J(t,x,\theta).$$

One of the challenges in solving the optimal stopping problem lies in characterizing the value function v, which represents the expected payoff or utility corresponding to the optimal stopping strategy. Proving a DPP, this problem is associated with an HJB equation of the following form

$$\min\{\partial_t v + Lv, v - g\} = 0,$$

with the operator L defined as in (0.0.1).

This PDE is commonly referred to as the *obstacle problem*, which governs the determination of the optimal stopping strategy. The variable's domain is partitioned in two: the continuation region and the stopping one. The continuation region encompasses points that fulfill the condition $v > g$. When the state process $X^{t,x}$ falls within this region, the evolution of the value function follows the linear PDE $\partial_t v + Lv = 0$, indicating that it is sub-optimal to stop the process. Conversely, the stopping region is characterized by the constraint $v = g$. As soon as the state process satisfies this condition, it becomes optimal to halt the dynamics.

The optimal stopping problem finds a significant connection to the pricing of American options in financial markets (see, *e.g.*, [84, 132]). American options grant the holder the right to exercise the option at any time before its expiration date. Determining the optimal time to exercise an American option corresponds to solving an optimal stopping problem.

Area 2: Branching diffusion processes

Continuous-time branching diffusion processes belong to the class of continuous-time branching particle systems, wherein the spatial motion, representing the feature of interest, evolves according to a continuous diffusion process. Specifically, these processes are described by a continuous-time stochastic process, representing the population size or density at each time point. As discussed in [54, 68, 72, 133], a branching diffusion process is characterized by three key components: the spatial motion, the branching rate, and the branching mechanism. We refer to [5, 71, 116] for monographs in this field.

The spatial motion that governs the behavior of each particle of the population in space is a diffusion process of the form (0.0.2), where X denotes the position of the particle. The branching rate γ encodes the average lifetime of each particle in the system. It influences the rate at which particles give rise to offspring. Finally, the branching mechanism Φ governs the random number of particles generated when a death event takes place. The probability generating function of the branching mechanism is given by $\Phi(s) = \sum_{k\geq 0} p_k s^k$ for $s \in [0, 1]$, where p_k represents the probability of producing k offspring upon death.

These continuous-time branching diffusion processes provide a probabilistic framework for understanding population dynamics, particularly in biology. They have found applications in various fields, including ecology, epidemiology, and genetics. In *ecology*, this modeling is used to analyze the growth and spread of populations. They help in understanding the dynamics of species populations, species interactions, and the influence of environmental factors on population patterns (see, *e.g.*, [99, 97, 120]). In *epidemiology*, they are used to study the spread and control of infectious diseases, analyze disease transmission dynamics, estimate epidemic thresholds, and evaluate intervention strategies. They provide insights into the impact of various factors, such as contact rates, transmission probabilities, and spatial movement, on epidemic outcomes (see, *e.g.*, [110]). In *genetics*, they are used to investigate the evolution and propagation of genetic traits within populations, capturing genetic drift, mutation, and selection processes, providing a probabilistic framework to study genetic diversity and the spread of advantageous or deleterious alleles (see, *e.g.*, [35, 36]).

Representation of branching diffusion processes

Continuous-time branching diffusion processes can be described using two distinct approaches. The first approach involves considering their associated martingale problem in the space of right-continuous, left-limit (càdlàg) paths in the space of finite measures on $\mathbb{R}^d$. This formulation enables a rigorous mathematical treatment, providing valuable insights into the stochastic behavior of the population (see, *e.g.*, [9, 71, 72]).

The second approach employs Galton–Watson random trees to represent continuous-time branching diffusion processes. In this representation, these processes are modeled as real processes indexed over a Galton–Watson tree. This structure aptly captures the branching nature of the population, focusing on the genealogy that originates the considered population (see, *e.g.*, [114, 115])

The martingale problem formulation provides a comprehensive understanding of the stochastic dynamics of continuous-time branching diffusion processes. It enables the analysis of various statistical properties, including population size distributions, extinction probabilities, and growth rates, offering valuable insights into the system's underlying behavior.

Furthermore, the martingale problem formulation facilitates the examination of the scaling limits of these processes, allowing to study their behavior as the population size becomes large or small. However, this formulation does not explicitly reveal the underlying genealogy or ancestral relationships between individuals in the population.

While the martingale problem approach is powerful for studying the statistical aspects of continuous-time branching diffusion processes, the Galton–Watson tree representation is better suited when the major interest is exploring the genealogical structure of the population. The tree structure explicitly shows the ancestral connections between individuals and their offspring, providing a more intuitive understanding of the branching process and its implications for the population's evolution over time. Together, these two approaches complement each other, providing a comprehensive view of the dynamics and characteristics of continuous-time branching diffusion processes.

Ulam–Harris–Neveu notation

One effective strategy to graphically represent the genealogy in branching diffusion processes is using the Ulam–Harris–Neveu notation. This tool has been introduced to visualize the lineage relationships between individuals in a population as they undergo branching events. This notation simplifies the study of the evolution of branching diffusion processes and understanding how the offspring inherit traits and characteristics from their ancestors over time. It provides an intuitive and concise way to represent the complex genealogy of a population and is widely used in various fields, including genetics, ecology, and epidemiology.

In Ulam–Harris–Neveu notation, the set of labels $\mathcal{I}$ is defined as the union of sets $\mathbb{N}^n$ for all $n \geq 0$, where $\mathbb{N}^0 = \{\varnothing\}$. Consider the concatenation of two particles $i = i_1 \cdots i_p$ and $j = j_1 \cdots j_q$ is denoted by $ij = i_1 \cdots i_p j_1 \cdots j_q$. The special label $\varnothing$ is considered to be the *mother* of all particles and the neutral element for this operation, *i.e.*, $i\varnothing = \varnothing i = i$ for any $i \in \mathcal{I}$.

Using this semigroup operation, we can easily describe branching events. When a particle $i \in \mathcal{I}$ dies giving birth to k children, we assign them the labels $i0, \dots, i(k-1)$. This point of view portrays the ordering of being an ancestor to a particle with the following partial order relation in $\mathcal{I}$: for particles $i, j \in \mathcal{I}$, we denote $i \preceq j$ (resp. $i \prec j$) if there exists $\ell \in \mathcal{I}$ (resp. $\ell \in \mathcal{I} \setminus \{\varnothing\}$) such that $j = i\ell$.

Being countable, we enable the discrete topology on $\mathcal{I}$. Therefore, branching processes can now be seen as solutions to a martingale problem in the space of finite measures on $\mathcal{I} \times \mathbb{R}^d$. This elegant framework combines the two previous approaches, incorporating their respective advantages. The component on $\mathcal{I}$ immediately yields the associated Galton–Watson tree defining the genealogy. Meanwhile, the martingale problem formulation enables a more accessible mathematical treatment.

Superprocesses

Despite their straightforward formulation, the analysis of branching particle systems can present significant challenges. This complexity arises from potential intricate inter-dependencies among particles, especially when dealing with a large number of them. As a result, there is a strong motivation to explore scaling limits. This perspective offers valuable insights and emphasizes essential aspects of the model. Scaling approximations aid in comprehending the system's behaviors and provide a deeper understanding of its underlying dynamics.

Superprocesses and super Brownian motion, in particular, emerge as the scaling limits of branching diffusion processes with a critical branching mechanism, *i.e.*, $\sum_{k\geq 0} p_k k = 1$. This critical condition ensures a balance in the branching rate, resulting in fascinating scaling properties. Our focus on these limit objects aims to adopt a synthetic perspective on the macroscopic limit, thereby illuminating their collective behaviors and facilitating the analysis of real-world applications.

The rigorous proof of these results relies on the convergence of martingale problems. The martingale problem formulation allows for a precise mathematical treatment, ensuring that the limiting process captures the key features of the branching diffusion processes accurately. Some examples of this line of reasoning can be found in [54, 68, 72, 133, 140, 141].

Superprocesses and super Brownian motion have versatile applications across multiple scientific and practical domains. In ecology, they are utilized to model the spatial distribution of species and study the spread of epidemics (see, *e.g.*, [99, 109]). In finance, these processes are applied in option pricing, portfolio optimization, and modeling financial markets with interactive agents (see, *e.g.*, [123]). Additionally, in population genetics, superprocesses offer valuable insights into understanding genetic diversity and the evolution of traits within populations (see, *e.g.*, [75]). Their widespread applications demonstrate their significance in diverse fields, making them powerful tools for analyzing complex systems and phenomena in nature and society.

Furthermore, it is essential to emphasize that the dual process of superprocesses plays a fundamental role in the study of these stochastic systems. This process emerges from taking the Laplace transform for this class of processes, leading to the following quadratic PDE, known as the *dual equation*,

$$\frac{\partial}{\partial t} u = b(x)^\top Du + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x) D^2 u\right) - \gamma u^2.$$

This transformation converts the intricate branching and interaction dynamics into a PDE framework, offering a complementary perspective on the superprocess. The study of solutions to this PDE gives information on the behavior of the superprocess.

Conversely, this connection between this nonlinear, second-order dual PDE and the class of superprocesses provides a probabilistic representation for a new category of nonlinear PDEs (see, *e.g.*, [59]). This realization has led to the application of these processes in generalizing the standard Feynman–Kac formula. By leveraging the probabilistic perspective offered by the dual process, we gain insights into solving a wider range of nonlinear PDEs.

Area 3: McKean–Vlasov dynamics

Our interest lies in examining controlled dynamics that emerge as scaling limits of controlled processes. Our particular focus is on scenarios where the resulting process is infinite-dimensional, notably in the context of measure-valued dynamics. Instances where probabilistic methodologies are employed to establish controlled measure-valued dynamics as scaling limits are found in fields such as Mean Field Control (MFC) and Mean Field Games (MFG). These fields serve as extensions of the McKean–Vlasov dynamics, showcasing how probabilistic tools contribute to the definition and analysis of broader and more complex controlled systems.

The McKean–Vlasov dynamics describe the evolution of a system of particles in which each particle interacts with the collective behavior of the entire population. Formally, consider a large number of particles, indexed by i, whose dynamics follow a stochastic process. The dynamics of each particle depend on the empirical measure of the entire particle system. This coupling between individual and collective behavior leads to a self-consistent evolution, where the influence of a single particle on the population is determined by the average effect of all other particles.

The McKean–Vlasov dynamics can be expressed by the following SDE

$$dX_t = b(X_t, \mu_t)dt + \sigma(X_t, \mu_t)dB_t, \tag{0.0.10}$$

where X represents the state of the representative particle in the considered system and B is Brownian motion in $\mathbb{R}^d$. The crucial feature of (0.0.10) lies in the dependence of the drift and diffusion terms on the mean field $\mu_t = \mathcal{L}(X_t)$, which describes the law of X and captures the aggregated information of the entire particle system.

The coefficients b (resp. σ) are functions from $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ to $\mathbb{R}^d$ (resp. $\mathbb{R}^{d\times d}$), where $\mathcal{P}(\mathbb{R}^d)$ denotes the set of probability measures on $\mathbb{R}^d$. This space is generally restricted to $\mathcal{P}^2(\mathbb{R}^d)$, the subset of probability measures with a finite second-order moment, equipped with the Wasserstein 2-distance W_2

$$W_2(\mu, \nu) = \left(\inf_{\pi \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \pi(dxdy) \right)^{1/2}, \quad \text{for } \mu, \nu \in \mathcal{P}^2(\mathbb{R}^d),$$

This distance is a metric used to quantify the similarities and dissimilarities between two probability distributions in a metric space. It gives an evaluation of the cost required to transform one probability measure into another, where the cost is determined by the transportation distance in the underlying metric space.

The Wasserstein distance is of paramount importance in the theory of McKean–Vlasov dynamics. In fact, unlike the Prokhorov distance (see, *e.g.*, [20]), another common distance used in the space of probability measures, the Wasserstein distance is easier to manipulate both theoretically and computationally (see, *e.g.*, [15, 49]). These advantages are the reasons why it is a preferred choice for this context.

The study of equations like (0.0.10) has pushed the development of differential calculus over spaces of probability measures. This field, known as optimal transport and calculus of variations over probability measures, involves notions of gradient, derivative, and optimization over probability measures. The concepts of flat and Lions' derivatives have emerged as essential tools in this context, enabling the analysis of solutions to the McKean–Vlasov dynamics and the associated MFC and MFG problems. We refer to [28, 30, 31, 32, 117] for detailed descriptions of these results.

The existence and the uniqueness of solutions to (0.0.10) can be established by generalizing standard proofs for SDE in this setting. In particular, suppose b and σ Lipschitz, *i.e.*, there exists a constant $C > 0$ such that

$$|b(x,\mu) - b(y,\nu)| + |\sigma(x,\mu) - \sigma(y,\nu)| \leq C(|x-y| + W_2(\mu,\nu)),$$

and $X_0 \in \mathcal{P}^2(\mathbb{R}^d)$. Then, for any $T > 0$, there exists a unique strong solution to (0.0.10) on $[0,T]$. This provides a solid mathematical foundation for exploring emergent phenomena and collective behavior in complex systems with a vast number of interacting agents.

Propagation of chaos

Propagation of chaos is a key phenomenon observed in the McKean–Vlasov dynamics. It represents the counterpart of the law of large numbers for interacting particle systems, describing the behavior of the empirical measure as the number of particles N tends to infinity. Consider the systems of N stochastic differential equations in $\mathbb{R}^d$ such that $(X_0^{N,i})_{i=1,\dots,N}$ are i.i.d. and

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)dB_t^i,$$

where the $(B^i)_{i=1,\dots,N}$ are i.i.d Brownian motion, and $\mu_{X_t}^N$ is the empirical measure associated with $(X_t^{N,i})_{i=1,\dots,N}$ defined by $\mu_t^N(dx) = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ where δ is the Dirac measure. The propagation of chaos is the property that, under suitable conditions, as N tends to infinity, the empirical measure μ_t^N converges weakly to $\mathcal{L}(X_t)$, where X_t the solution of (0.0.10) with $\mu_t = \mathcal{L}(X_t)$.

This phenomenon showcases how the particles in the system interact collectively, giving rise to mean field behavior. As particles grow, the empirical measure increasingly approximates the mean field behavior, resulting in a self-consistent, deterministic evolution. This remarkable property has rendered it applicable in various domains, including economics, finance, crowd dynamics, and epidemiology. In particular, its generalizations, such as mean field control and mean field games, greatly contribute to its broad impact. These modifications have made (0.0.10) a versatile and powerful mathematical tool, enabling the understanding of collective behaviors, the derivation of optimal strategies, and the analysis of emergent phenomena in large-scale systems. Its broad scope and effectiveness make it an indispensable framework for addressing complex challenges across different fields of study.

Mean field control and mean field games

MFC and MFG are extensions and generalizations of McKean–Vlasov-type dynamics, providing two distinct developments that combine this dynamic framework with control theory. In these contexts, agents not only interact with each other through their collective behavior, as observed in (0.0.10), but also can optimize their actions in response to the entire population's behavior. MFC and MFG offer a deeper understanding of how control decisions and strategic interactions among agents influence the overall dynamics of large-scale systems, making them powerful tools for analyzing complex decision-making scenarios.

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual conditions and supporting a Brownian motion B, we consider the set $\mathcal{A}$ of $\mathbb{F}$-progressively measurable stochastic processes $\{\alpha_t\}_{t\geq 0}$ valued in A. For a fixed horizon $T > 0$, the evolution of the state process $X^{t,x,\alpha}$ is described by a McKean–Vlasov type SDE as follows

$$dX_s = b(X_s, \mu_s, \alpha_s)ds + \sigma(X_s, \mu_s, \alpha_s)dB_s, \qquad (0.0.11)$$

with $X_t = x$ for a fixed initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ and a control $\alpha \in \mathcal{A}$. The functions

$(b, \sigma) : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times A \to \mathbb{R}^d \times \mathbb{R}^{n\times d}$ are deterministic continuous functions Lipschitz in (x, μ) uniformly in a. Consider now $f : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times A \to \mathbb{R}$ and $g : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}$. For a flow of probability measures $\mu = \{\mu_t\}_{t\in[0,T]}$, we define the cost

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,\alpha}, \mu_s, \alpha_s)ds + g(X_T^{t,x,\alpha}, \mu_T)\right]. \tag{0.0.12}$$

analyzing how to link the behavior of X with that of μ and how to consider optimization with respect to the control α produces these two theories.

Introduced independently in [111, 112, 113] and [90, 91, 92, 93], the objective for the MFG problem is to derive a *Nash equilibrium*. This means that no agent can unilaterally improve the related reward given the collective behavior of others. In this setting, the objective is to comprehend the strategic interactions among agents and determine a Nash equilibrium for a number N of agents and then study the limiting behavior as N tends to infinity. Each agent strives to optimize their utility while accounting for the collective impact of all agents on the system. This setup precisely resembles that of a stochastic game, where two adversarial parties influence each other's strategies and adopt opposing behaviors to reach an equilibrium. In this context, these roles are taken on by a single representative player and the whole population. While this agent is affected by the global dynamics, she also represents the general behavior within her population.

From a mathematical perspective, this is described into two steps: firstly, formulating an optimization problem, and secondly, solving a fixed point problem as outlined below.

(i) For a fixed deterministic flow $\mu = \{\mu_t\}_{t\geq 0}$ of probability measures, solve the stochastic control problem

$$v(t, x) = \inf_{\alpha\in\mathcal{A}} J(t, x, \alpha),$$

with J as in (0.0.12) and X satisfying (0.0.11).

(ii) Find a flow $\mu = \{\mu_t\}_{t\geq 0}$ such that $\mathcal{L}(\hat{X}_s^{\mu}) = \mu_s$ for all $s \in [t, T]$, being $\hat{X}^{\mu}$ a solution of the above optimal control problem.

Considering an N-player game, as done for the approximation of the standard McKean–Vlasov dynamics, having this point of view corresponds to studying the convergence of the Nash equilibria for the N player game, as N goes to infinity. Here a propagation of chaos result is used to prove this convergence rigorous.

In contrast, MFC problems direct their attention towards optimizing the actions of individual agents while accounting for the collective impact of the entire population on their dynamics. This approach begins with the assumption that the agents involved in the optimization process are cooperative and rational decision-makers, exhibiting the same behavior. Consequently, the optimization involves an infinite number of participants. Using the identical behavior of each agent, one can reduce this infinite minimization to a single representative participant's perspective. By doing so, the formulation of MFC problems breaks down the macroscopic problem into a microscopic one. This point of view gives insights into the overall system's behavior emerging from interactions and decisions of each component, revealing the connection between the global and local dynamics of the mean field model.

Mathematically speaking, this means that (0.0.11) becomes

$$dX_s = b(X_s, \mathcal{L}(X_s), \alpha_s)ds + \sigma(X_s, \mathcal{L}(X_s), \alpha_s)dB_s,$$

and we face the following optimization problem under these dynamics

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^T f(X_s, \mathcal{L}(X_s), \alpha_s) ds + g(X_T, \mathcal{L}(X_T))\right],$$

The solutions to both problems offer approximations of equilibrium states for large populations of individuals with mean field-type interactions and objective functions. The distinctions between these equilibrium concepts are nuanced and hinge on how the optimization component is formulated in the equilibrium model. It is important to note that taking fixed points and optimizing are not commutative operations, but in certain instances, the optimal trajectories of a McKean–Vlasov type control problem can be derived from the solution of a mean field game, possibly influenced by different coefficients. This demonstrates the interplay between the two frameworks and highlights their relevance in modeling complex systems with interacting agents. To know more on this connection, see, *e.g.*, [31, Section 6.2 and Section 6.7].

These formulations have encountered extensive applications in various fields. In economics, these theories have been used to study the dynamics of financial markets, where agents' strategic interactions influence market behavior (see, *e.g.*, [18]). In traffic management, they aid in optimizing traffic flow and route planning in urban environments(see, *e.g.*, [77]). In epidemiology, they have been used in understanding the transmission dynamics of epidemics, optimizing intervention strategies, and predicting the overall behavior of the disease in a community (see, *e.g.*, [6]). In social sciences, mean field games are applied to analyze the spread of information and adoption of behaviors in social networks (see, *e.g.*, [46]). Moreover, in the field of Deep Learning, this theory finds practical applications in establishing a mathematical foundation for interpreting certain types of neural networks (see, *e.g.*, [89, 143]). Additionally, mean field control problems find applications in energy and power systems, optimizing energy consumption and power generation in smart grids (see, *e.g.*, [1, 2, 3, 57]). The versatility and robustness of mean field games and mean field control make them valuable tools in modeling and understanding complex systems with large populations of interacting agents.

Existing literature

We now briefly focus on some of the most important examples that have emerged in the exploration of the control of branching diffusion processes.

Controlled branching processes were first introduced in [152], where the author bases their modeling on a topological sum of Euclidean space. Within this framework, the influence of the control on the dynamics is confined solely to the drift of spatial movement, and the control space is assumed to be compact. In this setting, any particle can be influenced by any other living particle, without imposing a specific structure on the nature of these interactions. Similarly, the running cost exhibits a remarkable level of generality. These components lead to an equivalently complex differential characterization.

Narrowing down the scope to a more specific context allowed for a more in-depth analysis, as done in [125]. In this article, the author pursues a deeper investigation by concentrating on a particular cost function, defined as a product of functions valued in particles of the population at the terminal time. To explore this aspect, controlled branching processes are used as a probabilistic tool, specifically to examine a particular group of parabolic Bellman equations. In this study, the control, still restricted within a compact set, influences both the drift and volatility of the diffusion of each particle. As a result, an HJB equation is established for the value function, characterized by its unique (viscosity) solution.

In [42], the author extends the previous research line. The controlled processes are described as measure-valued processes, and the author introduces a Ulam–Harris–Neveu labeling to explicitly represent the genealogy of the particles. To provide a strong formulation for the controlled branching processes, a collection of Brownian motions and Poisson random measures, indexed by these labels, are utilized. This allows for dynamics where drift, volatility, branching rate, and branching mechanisms are not only controlled but also dependent on the position of each particle. Moreover, although these coefficients are assumed to be bounded, the control space is no longer necessarily restricted to be compact.

The coupling of dynamics through the control introduced in [42], together with the multiplicative-type cost function as in [125], leads to an interesting branching property, effectively transforming the problem into a finite-dimensional one, reducing the problem from a macroscopic optimization to a microscopic one over each particle. Leveraging the differential properties of the Euclidean space where each particle is defined, a PDE characterization of the value function is obtained. This framework enables a comprehensive exploration of the system with intricate control dependencies.

An instance of a study that combines the branching diffusion framework with the mean field approach is presented in [44], where the authors introduce scaling limits that deviate from the dynamics of superprocesses. Beginning with the MFG approach, the authors adapt it to the branching setting using a PDE approach. They establish a relaxed formulation for the controlled dynamics under consideration and employ it rigorously to extend the MFG concept to branching particle systems. This method paves the way to the proof of the existence of solutions for the stochastic game under consideration and establishes an approximate Nash equilibrium for large population games. By incorporating both the branching diffusion and mean field perspectives, this study presents a comprehensive framework that provides valuable insight into stochastic processes and population games.

Contributions

In this thesis, our objective is to continue this line of research by combining tools from the previously discussed disciplines. We aim to uncover underlying structures and identify how these methodologies complement and enrich the properties and outcomes of each other. Our contributions can be categorized into two main macro-areas:

- Application of conventional stochastic control problems to branching processes.
- Development of a novel approach for stochastic control theory for branching processes, under symmetry assumption, and analysis of its scaling limit counterpart.

Stochastic control versus branching diffusion processes

The first part of the study centers on investigating a stochastic target problem and optimal stopping for branching processes. Leveraging the branching property as a guiding principle, we aim to transform the analysis, typically infinite-dimensional due to the nature of the stochastic processes studied, into a finite-dimensional approach. To achieve this, we combine cost functions that respect the problem's symmetry.

We observed that in both cases, the genealogy's structure, considering cost functions dependent on the label of the associated particle, fosters a profound interaction between optimization and branching. This interaction is particularly evident in the resulting HJB equations found in both chapters. Solutions to these two problems involve a system of PDEs indexed on the label set, whose unique viscosity solution is shown to be the value function.

Stochastic target problem for branching diffusion processes

In Chapter 1, taken from the article [102], the modeling approach relies on a strong formulation of branching processes as measure-valued processes. Within this framework, the branching parameters are considered to be independent of both spatial position and control. Introducing a coupling between the observed population and the target of interest, we establish a linked system that encompasses both entities. The chosen target reflects the underlying symmetry of the problem, where the objective is to super-replicate the entirety of the process with another branching population that synchronizes with the one under analysis.

We use a DPP approach to define the value function for the stochastic target problem under consideration. The aforementioned symmetry results in a geometric DPP and a branching property, which reduces the complexity of the problem. Here, the value function that spans the entire population is proven to be equal to the maximum of value functions on each particle within the population.

The lack of smoothness of the maximum function is further amplified by the nature of the resulting HJB equation. Specifically, the PDE that defines this problem imposes restrictions on the gradient of the value function. This intrinsic non-linearity, commonly found in variational inequalities with gradient constraints, presents an additional challenge when attempting to approximate the value function using solutions from regularised problems.

To overcome this difficulty, we turn to the theory of viscosity solutions, providing a tailored definition suitable for the problem at hand. Notably, the explicit dependence of reward functions on the particle index introduces a bound on the test functions with respect to the indexed, which is effectively treated as a state variable of the problem. This approach allows us to prove a comparison theorem, gaining deeper insights into the properties and behavior of the stochastic target problem. We subsequently characterize the value function, relying on Ishii's lemma, a result that traditionally pertains to HJB equations in finite-dimensional spaces. Therefore, we can observe how the advantageous branching property plays a pivotal role in resolving the initial problem.

Optimal stopping of branching diffusion processes

In Chapter 1, we employ the concept of the stopping line, which serves as the counterpart to stopping time in branching dynamics. By adopting a stopping criterion tailored to the particle's evolution with this object, we align with a microscopic perspective that focuses on comprehending the decision-making process of each individual within a reward framework linked to the specific genealogical lineage under consideration. The focus of this analysis lies in modeling processes with values in $\mathbb{R}^d$ indexed on a Galton–Watson tree. Although this formalism may seem more complex to manipulate, it offers a more precise examination of single-particle behavior.

We examine a reward function that operates on particles in a product-like manner and seek to establish the optimal stopping point for each genealogical lineage. This form of cost function bears similarities to the optimal control problem previously explored case in the aforementioned works. More specifically, our examination is rooted in a hypothesis regarding the vanishing nature of rewards as a function of the number of generations. We focus on scenarios where reward functions depend on the particle index and approach zero as the number of generations escalates. This characteristic enables us to establish regularity outcomes concerning the value function, including attributes such as polynomial growth and global continuity.

This framework makes possible the discovery of a novel adaptation of the DPP, one that aligns closely with the branching behavior inherent in these stochastic processes. For a fixed stopping line, this result reveals a dynamic interplay between two branching components: the already halted portion and the one that remains capable of evolving over time. Moreover, with the use of the regularity property of the value function, proving this result is simpler than its analog in the preceding chapter as no measurable selection result is needed. This is facilitated through the construction of stopping lines, where the associated reward functions prove to be ϵ-optimal.

This competitive interaction among different population segments draws an immediate parallel to the classical obstacle problem, a scenario that divides space into two distinct regions. This very division is inherited by the resulting HJB equation, unveiling two distinct behaviors. The first corresponds to the stopping region, signifying that when the value function aligns with a particle's reward function, halting becomes optimal. The second is related to the continuation region. Here, a new element compared to the classical setup emerges with a coupling between the value functions associated with a specific particle and its direct descendants. This inter-dependency adds a level of complexity to the system and stems from the more intricate dynamics of this class of processes.

Similarly to Chapter 1, this intricate interrelation leads to delving into the realm of viscosity solution theory. With a tailored definition of viscosity solutions, accounting for the radius of convergence of the power series linked to the branching mechanism, we achieve two significant outcomes. Firstly, we establish that the value function is a viscosity solution for the specific system of PDEs under consideration. Secondly, we prove a comparison principle that capitalizes, once again, on the vanishing nature of the reward functions as the generation grows.

Controlled branching diffusion processes under symmetry assumptions

In the second part of the thesis, our focus was to develop a stochastic control theory for branching processes under the assumption of symmetry of the cost function on particle labeling. This approach allows us to explore and understand the dynamics of branching processes under more general and flexible settings, where the control and optimization aspects are not tied to the individual particle identities, but only to their representative behavior. This adaptation enables us to develop a framework that we later use to investigate the scaling limit of these objects, originating the class of controlled superprocesses.

Relaxed formulation for controlled branching diffusion processes

In Chapter 3, taken from the article [128], an extension of controlled branching processes is explored. This novel construction introduces an interdependence among particles, governed by their empirical population measure. This framework facilitates the investigation of a novel category of processes, which, akin to the N-particle approximation within the MFC, captures symmetric inter-particle correlations, simulating the influence of the overall population on individual dynamics. Additionally, we shift our attention towards cost functions that, distinct from the preceding two chapters, eschew explicit dependency on particle indices.

In this context, the process can be regarded as taking values in the space of càdlàg paths on finite measures over $\mathbb{R}^d$. Consequently, while direct genealogical visualization may be sacrificed, a more streamlined and manageable perspective emerges.

This chapter's primary aim revolves around an in-depth exploration of diverse definitions for controlled branching processes, which extend the concept of strong control. Starting from a fitting definition of relaxed control, akin to diffusion processes, we proceed to establish the categories of controls: natural control, weak control, and control rules. Our initial objective encompasses an extension of classical findings, revealing the intricate interplay and distinctions amongst these classes.

In pursuit of our objective to examine processes characterized by linear growth drift in the spatial variable, after having defined the class of strong controls, our initial step involves deriving moment estimates for these processes. These estimates subsequently enable us to consider cost functions that adhere to specific coercivity conditions. Leveraging these conditions, we proceed to formulate the problem related to strong controls, ensuring its well-posedness.

We establish the problem of relaxed control, a concept arising from our consideration of controls no longer reliant on particle indices, but rather contingent upon particle positions. Within this framework, we define the class of relaxed controls, as solutions to a martingale problem, with initial values in the subspace of finite Dirac sums within $\mathbb{R}^d$, while adhering to specific conditions on the nature of control. This control is conceived as a finite measure on the space $[0,T] \times \mathbb{R}^d \times A$, where $T > 0$ denotes the finite horizon and A denotes the control space, with its projection onto the first component representing the Lebesgue measure.

Notably, we observe a relationship akin to the case of diffusion processes, where the class of strong controls is encompassed within this broader classification. Furthermore, we proceed to establish a representation theorem, enhancing our ability to manipulate the martingale problem central to the definition of relaxed controls.

With this definition, our next stride entails delving into the relationship between the two formulations of the optimization problem, proving their equivalence. The initial stride towards this objective is to introduce an intermediary category - the realm of natural controls. These controls emerge as relaxed controls defined within the canonical space, characterized by the filtration generated by the canonical process. Within this framework, we substantiate the proposition that for any given relaxed control, a corresponding natural control can be associated, bearing a cost no less than that of the initial relaxed control.

Advancing towards establishing equivalence, the second phase entails a focus on weak controls. This setting is introduced to encapsulate the embedding relationship between strong and relaxed controls as discussed earlier. Subsequently, we proceed to obtain that, subject to a standard usual assumption in stochastic control problems of the Filippov type, the optimization problem involving natural controls can be confined to weak controls without augmenting the value function. Ultimately, our journey culminates in affirming the equivalence between weak and strong controlsthe pivotal element in finalizing the equivalence between relaxed and strong controls.

In light of this newly introduced formalism, we now turn our attention to an intermediary classification, nestled between relaxed and natural controls, termed as *control rules*. This strategic move enables us to reformulate the optimization task into a problem over a subset of probability measures, satisfying certain conditions. Through a strategic utilization of the topological properties of the space of probability measures, we are able to show that the coercive traits associated with cost functions re-frame the original optimization task as the minimization of a lower semi-continuous function within a compact space. This elegant restructuring not only solidifies the existence of optimal values and controls but also furnishes robust theoretical guarantees for the existence of optimal solutions within this category of processes.

Controlled superprocesses

In Chapter 4, taken from the article [127], we employ the tools introduced in the preceding chapter to delve into the scaling limits of the controlled branching diffusion processes. Initially, we present the definition of this novel category of processes known as the *controlled superprocesses*, utilizing a martingale problem framework. With this definition, our primary objective is to establish both the uniqueness in the law of the analyzed martingale problem and the existence of its solutions. The latter goal is achieved by considering the scaled versions of the processes introduced earlier in the preceding chapter, where their existence stems from a strong construction. By extending the Aldous criterion to the controlled context, we validate that for any given control, there exists at most a single probability distribution satisfying the martingale problem that defines the controlled superprocesses category.

Within this novel theoretical framework, our attention turns towards tackling a control problem incorporating both a running cost function and a terminal cost function. This approach lays the foundation for a DPP, accompanied by a differential characterization of the cost function.

Prior to the DPP, we revisit the weak formulation, introduced earlier for controlled branching diffusion processes, for controlled superprocesses, as outlined in Chapter 3. This approach facilitates the treatment of the control problem as it is seen as a probability measure that adheres to specific conditions. Notably, as these probability measures are defined over a Polish space, we retrieve two crucial properties concerning the set of controls under consideration: stability through conditioning and stability through concatenation. These pivotal characteristics serve as the foundation for extending the classical outcomes of measurable selection, culminating in the derivation of the associated DPP.

Given that the value function is defined within the space of finite measures over $\mathbb{R}^d$, we first focus on the differential properties of this space. This entails exploring the flat derivative and the intrinsic derivative, concepts originally developed in the MFG-MFC theory for probability measures and adeptly adapted to the context of finite measures. Leveraging these derivatives, we capitalize on the density theorems on cylindrical functions in the space of continuous functions over finite measures.

In particular, we consider the class of cylindrical functions with specific degrees of differentiability, dense in the space of continuous and sufficiently differentiable functions with respect to these derivative notions. This pivotal step makes us extend the initial martingale problem and re-frame it through the lens of these differential concepts obtaining a generalized martingale problem. This leads us to define the HJB equation associated with this control problem. Therefore, we can give a verification theorem and characterize the optimal solution strategies for the considered control problem.

The concluding segment of this chapter culminates in the presentation of illustrative examples where we can exhibit regular solutions. Considering the problem's inherently infinite-dimensional nature, as it unfolds within the domain of finite measures, establishing the existence of smooth value functions proves to be challenging.

Hence, we assume that the coefficients governing the dynamics of controlled superprocesses are independent from the state measure, and undergo certain regularity conditions. By considering exponential-type cost functions, and adopting a methodology reminiscent of the approach taken in the initial two chapters, we uncover a branching property. This reduces the dimensionality of the control problem into a finite-dimensional optimization. This outcome bears resemblance to the MFC viewpoint, where global optimization is translated into optimizing the behavior of individual participants, representing the entire community to which they belong.

In fact, within the context of the aforementioned assumptions and capitalizing on the previously established verification theorem, optimizing the global dynamics can be re-framed as the optimization of finite-dimensional dynamics of a representative participant of the population under analysis. Computing the value function for a given measure simplifies taking the exponential of the integral of the solution of the preceding finite-dimensional HJB equation with respect to that particular measure. Furthermore, from the stated regularity of the coefficients, we attain the existence of classical solutions to the finite-dimensional HJB equation. This comprehensive framework culminates in the demonstration of a category of regular value function problems, all achieved without invoking the theory of viscosity solutions.

Outline of the manuscript

Each chapter is independent and self-contained. Therefore, the notation may vary from chapter to chapter. We summarize the contents of the chapters below.

– Chapter 1 is taken from [102] and is a joint work with Idris Kharroubi. It has been submitted for publication.

– Chapter 2 is a joint work with Idris Kharroubi.

– Chapter 3 is taken from [128] and it has been submitted for publication.

– Chapter 4 is taken from [127] and it has been submitted for publication.

All of these contributions seek to bridge the gap between these different theories. They are examples of a fruitful field of research, building upon well-established theories. By developing both global process perspectives and individual behavior considerations, we aim to explore more and more intricate dynamics to give a better representation of real-world scenarios.

Résumé détaillé

Sommaire du présent chapitre

Dans cette thèse, notre objectif est de combiner les outils des trois disciplines suivantes : contrôle stochastique, processus de branchement et dynamique de McKean–Vlasov. Nous visons à découvrir les structures sous-jacentes et à identifier comment ces méthodologies complètent et enrichissent les propriétés et les résultats des unes et des autres. Nos contributions peuvent être classées en deux macro-domaines principaux :

- Application des problèmes conventionnels de contrôle stochastique aux processus de branchement.
- Développement d'une nouvelle approche pour la théorie du contrôle stochastique pour les processus de branchement, sous l'hypothèse de symétrie, et analyse de sa contrepartie de limite d'échelle.

Contrôle stochastique et processus de diffusion branchants

La première partie de l'étude est centrée sur l'analyse d'un problème de cible stochastique et d'arrêt optimal pour les processus de branchement. En nous appuyant sur la propriété de branchement comme principe directeur, nous visons à transformer l'étude, typiquement infini dimensionnelle en raison de la nature des processus stochastiques considérés, en une approche finie dimensionnelle. Pour ce faire, nous combinons des fonctions de coût qui respectent la symétrie du problème.

Nous avons observé que dans les deux cas, la structure de la généalogie, qui considère des fonctions de coût dépendant de l'étiquette de la particule associée, favorise une interaction profonde entre l'optimisation et le branchement. Cette interaction est particulièrement évidente dans les équations Hamilton–Jacobi–Belman (HJB) trouvées dans les deux chapitres. Les solutions à ces deux problèmes impliquent un système d'EDP indexé sur l'ensemble des étiquettes, dont la solution unique de viscosité s'avère être la fonction valeur.

Problème de cible stochastique pour les processus de diffusion branchants

Dans le chapitre 1, extrait de l'article [102], l'approche de modélisation repose sur une formulation forte des processus de branchement en tant que processus à valeur de mesure. Dans ce cadre, les paramètres de brachement sont considérés comme indépendants de la position spatiale et du contrôle. En introduisant un couplage entre la population observée et la cible d'intérêt, nous établissons un système lié qui englobe les deux entités. La cible choisie reflète la symétrie sous-jacente du problème, où l'objectif est de super-répliquer l'intégralité du processus avec une autre population branchante qui se synchronise avec celle qui fait l'objet de l'analyse.

Nous utilisons une approche qui s'appuie sur le Principe de la Programmation Dynamique (PPD) pour caractériser la fonction valeur pour le problème cible stochastique considéré. La symétrie susmentionnée se traduit par une PPD géométrique et une propriété de branchement, ce qui réduit la complexité du problème. Ici, il est prouvé que la fonction valeur qui couvre l'ensemble de la population est égale au maximum des fonctions valeur sur chaque particule au sein de la population.

Le manque de régularité de la fonction maximale est encore amplifié par la nature de l'équation de HJB résultante. Plus précisément, l'EDP qui définit ce problème impose des restrictions sur le gradient de la fonction valeur. Cette non-linéarité intrinsèque, que l'on retrouve généralement dans les inégalités variationnelles avec des contraintes de gradient, représente un défi supplémentaire lorsque l'on tente d'approcher la fonction valeur à l'aide de solutions provenant de problèmes régularisés.

Pour surmonter cette difficulté, nous nous tournons vers la théorie des solutions de viscosité, en fournissant une définition adaptée au problème en question. Notamment, la dépendance explicite des fonctions de récompense par rapport à l'indice des particules introduit une limite sur les fonctions de test par rapport à l'indice, qui est effectivement traité comme une variable d'état du problème. Cette approche nous permet de prouver un théorème de comparaison et de mieux comprendre les propriétés et le comportement du problème de la cible stochastique. Nous caractérisons ensuite la fonction valeur en nous appuyant sur le lemme d'Ishii, un résultat qui se rapporte traditionnellement aux équations HJB dans des espaces de dimension finie. Ainsi, nous pouvons observer comment la propriété de branchement joue un rôle central dans la résolution du problème initial.

Arrêt optimal des processus de diffusion branchants

Dans le chapitre 2, nous utilisons le concept de ligne d'arrêt, qui sert d'équivalent au temps d'arrêt dans une dynamique branchante. En adoptant un critère d'arrêt adapté à l'évolution de la particule avec cet objet, nous nous alignons sur une perspective microscopique qui se concentre sur la compréhension du processus de prise de décision de chaque individu dans un cadre de récompense lié à la lignée généalogique spécifique considérée. Cette analyse se concentre sur la modélisation des processus avec des valeurs dans $\mathbb{R}^d$ indexées sur un arbre de Galton–Watson. Bien que ce formalisme puisse sembler plus complexe à manipuler, il permet un étude plus précis du comportement d'une particule.

Nous examinons une fonction de récompense multiplicative et cherchons à établir le critère d'arrêt optimal pour chaque lignée généalogique. Cette forme de fonction de coût présente des similitudes avec le problème de contrôle optimal précédemment exploré dans la literature. Plus précisément, nous nous concentrons sur des scénarios où les fonctions de récompense dépendent en plus de l'indice de particule et s'approchent de zéro lorsque le nombre de générations augmente. Cette caractéristique nous permet d'établir des propriétés tels que la croissance polynomiale et la continuité globale de la fonction valeur.

Ce cadre rend possible la découverte d'une nouvelle adaptation du PPD, qui s'aligne étroitement sur le comportement de branchement inhérent à ces processus stochastiques. Pour une ligne d'arrêt fixe, ce résultat révèle une interaction dynamique entre deux composantes de branchement : la partie déjà arrêtée et celle qui reste capable d'évoluer au fil du temps. De plus, grâce à l'utilisation de la propriété de régularité de la fonction valeur, la démonstration de ce résultat est plus simple que son analogue dans le chapitre précédent puisqu'aucune sélection mesurable n'est nécessaire. Ceci est facilité par la construction de lignes d'arrêt, où les fonctions de récompense associées s'avèrent être ϵ-optimales.

Cette interaction compétitive entre différents segments de population établit un parallèle immédiat avec le problème de l'obstacle classique, un scénario qui divise l'espace en deux régions distinctes. Cette même division est héritée par l'équation de HJB résultante, dévoilant deux comportements distincts. Le premier correspond à la région d'arrêt, ce qui signifie que lorsque la fonction valeur s'aligne sur la fonction de récompense d'une particule, l'arrêt devient optimal. Le second est lié à la région de continuation. Ici, un nouvel élément par rapport à la configuration classique émerge avec un couplage entre les fonctions valeur associées à une particule spécifique et ses descendants directs. Cette interdépendance ajoute un niveau de complexité au système et découle de la dynamique plus complexe de cette classe de processus.

Comme au chapitre 1, cette interrelation complexe conduit à plonger dans le domaine de la théorie des solutions de viscosité. Avec une définition adaptée des solutions de viscosité, tenant compte du rayon de convergence de la série génératrice liée au mécanisme de branchement, nous obtenons deux résultats significatifs. Premièrement, nous établissons que la fonction valeur est une solution de viscosité pour le système spécifique d'EDP considéré. Deuxièmement, nous prouvons un principe de comparaison qui capitalise, une fois de plus, sur la nature évanouissante des fonctions de récompense au fur et à mesure que la génération croît.

Processus de diffusion de branchement contrôlé sous des hypothèses de symétrie

Dans la seconde partie de la thèse, nous nous sommes attachés à développer une théorie du contrôle stochastique pour les processus de branchement sous l'hypothèse de symétrie de la fonction de coût sur l'étiquetage des particules. Cette approche nous permet d'explorer et de comprendre la dynamique des processus de branchement dans des contextes plus généraux et plus flexibles, où les aspects de contrôle et d'optimisation ne sont pas liés aux identités individuelles des particules, mais seulement à leur comportement représentatif. Cette adaptation nous permet de développer un cadre que nous utilisons par la suite pour étudier la limite d'échelle de ces objets, à l'origine de la classe des superprocessus contrôlés.

Formulation relâchée pour les processus de diffusion branchants contrôlés

Le chapitre 3, tiré de l'article [128], explore une extension des processus de branchement contrôlés. Cette nouvelle construction introduit une interdépendance entre les particules, régie par leur mesure de population empirique. Ce cadre facilite l'étude d'une nouvelle catégorie de processus qui, à l'instar de l'approximation des N particules dans le contrôle à champs moyen, capture les corrélations interparticulaires symétriques, simulant l'influence de la population globale sur la dynamique individuelle. En outre, nous portons notre attention sur les fonctions de coût qui, contrairement aux deux chapitres précédents, ne dépendent pas explicitement des indices de particules.

Dans ce contexte, les processus peuvent être considérés comme prenant des valeurs dans l'espace des chemins càdlàg sur des mesures finies sur $\mathbb{R}^d$. Par conséquent, bien que la visualisation généalogique directe soit sacrifiée, une perspective plus allégée et plus facile à maîtriser émerge.

L'objectif principal de ce chapitre tourne autour d'une exploration en profondeur de diverses définitions de processus de branchement contrôlés, qui étendent le concept de contrôle fort. En partant d'une définition appropriée du contrôle relâché, nous établissons les catégories de contrôle.

Notre objectif initial comprend une extension des résultats classiques, révélant l'interaction complexe et les distinctions entre ces catégories.

Dans la poursuite de notre objectif d'examiner les processus caractérisés par une dérive de croissance linéaire de la variable spatiale, après avoir défini la classe des contrôles forts, notre première étape consiste à dériver des estimations de moment pour ces processus. Ces estimations nous permettent ensuite de considérer des fonctions de coût qui adhèrent à des conditions de coercivité spécifiques. En nous appuyant sur ces conditions, nous formulons le problème lié aux contrôles forts, en veillant à ce qu'il soit bien posé.

Nous établissons le problème du contrôle relâché, un concept qui découle de notre considération des contrôles qui ne dépendent plus des indices de particules, mais plutôt des positions des particules. Dans ce cadre, nous définissons la classe des contrôles relâchés en tant que solutions à un problème de martingale, avec des valeurs initiales dans le sous-espace des sommes de Dirac finies dans $\mathbb{R}^d$, tout en adhérant à des conditions spécifiques sur la nature du contrôle. Ce contrôle est conçu comme une mesure finie sur l'espace $[0,T] \times \mathbb{R}^d \times A$, où $T > 0$ désigne l'horizon fini et A désigne l'espace de contrôle, en imposant que sa projection sur la première composante soit la mesure de Lebesgue.

Nous observons notamment une relation similaire au cas des processus de diffusion, où la classe des contrôles forts est englobée dans cette classification plus large. En outre, nous établissons un théorème de représentation qui nous permet de mieux manipuler le problème des martingales qui est au cœur de la définition des contrôles relâchés.

Avec cette définition, notre prochaine étape consiste à approfondir la relation entre les deux formulations du problème d'optimisation, en prouvant leur équivalence. La première étape vers cet objectif consiste à introduire une catégorie intermédiaire - le domaine des contrôles naturels. Ces contrôles apparaissent comme des contrôles relâchés définis dans l'espace canonique, caractérisé par la filtration générée par le processus canonique. Dans ce cadre, nous justifions la proposition selon laquelle à tout contrôle relâché donné peut être associé un contrôle naturel correspondant, dont le coût n'est pas inférieur à celui du contrôle relâché initialement considéré.

En progressant vers l'établissement de l'équivalence, la deuxième phase se concentre sur les contrôles faibles. Ce cadre est introduit pour encapsuler la relation d'intégration entre les contrôles forts et relâchés, comme nous l'avons vu précédemment. Ensuite, nous obtenons que, sous réserve d'une hypothèse standard dans les problèmes de contrôle stochastique de type Filippov, le problème d'optimisation impliquant des contrôles naturels peut être confiné à des contrôles faibles sans augmenter la fonction valeur. Finalement, notre voyage culmine dans l'affirmation de l'équivalence entre les contrôles faibles et forts - l'élément pivot dans la finalisation de l'équivalence entre les contrôles relâchés et forts.

À la lumière de ce nouveau formalisme, nous nous intéressons maintenant à une classification intermédiaire, nichée entre les contrôles relâchés et les contrôles naturels, appelée *règles de contrôle*. Ce mouvement stratégique nous permet de reformuler notre optimisation en un problème sur un sous-ensemble de mesures de probabilité, satisfaisant certaines conditions. Grâce à une utilisation stratégique des propriétés topologiques de l'espace des mesures de probabilité, nous sommes en mesure de montrer que les traits coercifs associés aux fonctions de coût transforment l'optimisation originale en la minimisation d'une fonction semi-continue inférieurement dans un espace compact. Cette reformulation élégante permet non seulement de montrer l'existence de valeurs et de contrôles optimaux, mais aussi de fournir des garanties théoriques solides pour l'existence de solutions optimales dans cette catégorie de processus.

Superprocessus contrôlés

Dans le chapitre 4, tiré de l'article [127], nous utilisons les outils introduits dans le chapitre précédent pour étudier les limites d'échelle des processus de branchement diffusif contrôlé. Dans un premier temps, nous présentons la définition de cette nouvelle catégorie de processus connue sous le nom de *superprocessus contrôlés*, en utilisant un cadre de problème de martingale. Avec cette définition, notre objectif principal est d'établir à la fois l'unicité de la loi du problème de martingale analysé et l'existence de ses solutions. Ce dernier objectif est atteint en considérant les versions réduites des processus introduits dans le chapitre précédent, où leur existence découle d'une construction forte. En étendant le critère d'Aldous au contexte contrôlé, nous validons que pour tout contrôle donné, il existe au plus une seule distribution de probabilité satisfaisant le problème de martingale qui définit la catégorie des superprocessus contrôlés.

Dans ce nouveau cadre théorique, notre attention se porte sur la résolution d'un problème de contrôle incorporant à la fois une fonction de coût de fonctionnement et une fonction de coût final. Cette approche jette les bases d'un PPD, accompagné d'une caractérisation différentielle de la fonction de coût.

Avant le PPD, nous revisitons la formulation faible, introduite précédemment pour les processus de diffusion branchants contrôlés, pour les superprocessus contrôlés, comme indiqué au chapitre 3. Cette approche facilite le traitement du problème de contrôle car il est considéré comme une optimisation sur des mesures de probabilité qui adhèrent à des conditions spécifiques. Notamment, comme ces mesures de probabilité sont définies sur un espace polonais, nous retrouvons deux propriétés cruciales concernant l'ensemble des contrôles considérés : la stabilité par conditionnement et la stabilité par concaténation. Ces caractéristiques essentielles servent de base à l'extension des résultats classiques de la sélection mesurable, qui aboutissent à la dérivation du PPD associée.

Étant donné que la fonction valeur est définie dans l'espace des mesures finies sur $\mathbb{R}^d$, nous nous concentrons d'abord sur les propriétés différentielles de cet espace. Cela implique l'exploration de la dérivée plate et de la dérivée intrinsèque, concepts développés à l'origine dans la théorie jeux à champs moyen pour les mesures de probabilité et habilement adaptés au contexte des mesures finies. En nous appuyant sur ces dérivées, nous capitalisons sur les théorèmes de densité des fonctions cylindriques dans l'espace des fonctions continues sur les mesures finies.

En particulier, nous considérons la classe des fonctions cylindriques avec des degrés spécifiques de différentiabilité, denses dans l'espace des fonctions continues et suffisamment différentiables par rapport à ces notions de dérivées. Cette étape cruciale nous permet d'étendre le problème de martingale initial grâce aux concepts différentiels, obtenant et ainsi un problème de martingale généralisé. Ceci nous amène à définir l'équation de HJB associée à ce problème de contrôle. Par conséquent, nous pouvons donner un théorème de vérification et caractériser les stratégies de solutions optimales pour le problème de contrôle considéré.

La dernière partie de ce chapitre se termine par la présentation d'exemples illustratifs où nous pouvons exhiber des solutions régulières. Compte tenu de la nature intrinsèquement infinie-dimensionnelle du problème, qui se déroule dans le domaine des mesures finies, l'établissement de l'existence de fonctions valeur régulières s'avère être un défi.

Nous supposons donc que les coefficients régissant la dynamique des superprocessus contrôlés sont indépendants de la mesure d'état et qu'ils sont soumis à certaines conditions de régularité. En considérant des fonctions de coût de type exponentiel, et en adoptant une méthodologie rappelant l'approche adoptée dans les deux premiers chapitres, nous découvrons une propriété de branchement. Ceci réduit la dimensionnalité du problème de contrôle en une optimisation à dimension finie. Ce résultat ressemble au point de vue du contrôle à champs moyen, où l'optimisation globale se traduit par l'optimisation du comportement des participants individuels, représentant l'ensemble de la communauté à laquelle ils appartiennent.

En fait, dans le contexte des hypothèses susmentionnées et en capitalisant sur le théorème de vérification précédemment établi, l'optimisation de la dynamique globale peut être reformulée comme l'optimisation de la dynamique à dimension finie d'un participant représentatif de la population analysée. Le calcul de la fonction valeur pour une mesure donnée simplifie la prise de l'exponentielle de l'intégrale de la solution de l'équation de HJB à dimension finie précédente par rapport à cette mesure particulière. De plus, grâce à la régularité des coefficients, nous obtenons l'existence de solutions classiques à l'équation de HJB à dimension finie. Ce cadre complet aboutit à la démonstration d'une catégorie de problèmes de fonctions valeur régulières, le tout sans invoquer la théorie des solutions de viscosité.

Chapter 1

A Stochastic Target Problem for Branching Diffusion Processes

Outline of the current chapter

This chapter corresponds to the paper [102], which has been submitted for publication. It is a joint work with Idris Kharroubi.

Abstract: We consider an optimal stochastic target problem for branching diffusion processes. This problem consists in finding the minimal condition for which a control allows the underlying branching process to reach a target set at a finite terminal time for each of its branches. This problem is motivated by an example from fintech where we look for the super-replication price of options on blockchain-based cryptocurrencies. We first state a dynamic programming principle for the value function of the stochastic target problem. Next, we shoow that the value function can be simplified into a novel function with the use of a finite-dimensional argument through a concept known as the branching property. Under wide conditions, this last function is shown to be the unique viscosity solution to an HJB variational inequality.

1.1 Introduction

The theory of optimal stochastic control has been extensively developed since the pioneering works in the 1950s decade. One reason for the growing attraction of this theory is the variety of its applications, such as physics, biology, economics, and finance.

In the last field, stochastic control theory appears to be a very natural tool as it provides solutions to the optimal portfolio choice issue. The need to control risks related to financial investments leads to new stochastic optimization problems. Here, one looks for the minimal initial endowment needed to find a financial strategy whose final position satisfies some given constraints. Such optimization problems are called optimal stochastic target problems and have been widely studied (see, *e.g.*, [22, 24, 25, 26, 145, 146]).

The classical stochastic control theory has also been developed for other kinds of stochastic processes such as branching diffusion processes. Those processes describe the evolution of a population of individuals with similar features concerning their dynamics and their reproduction. Branching processes were first studied by Skorohod [144] and Ikeda et al. [94, 95, 96], who provided Feynmann-Kac presentation of the solution to parabolic semi-linear Partial Differential Equations (PDE). Since those pioneering works, branching processes have been extensively studied in particular their scaling limits and the link with superprocesses (see, *e.g*, [54]). Recently, they were also used by Henry-Labordère et al. [87] for Monte Carlo-based numerical approximation of solutions to semilinear parabolic PDEs.

In the case where the branching processes are controlled, Üstünel [152] considers a finite horizon optimization problem. He restricted to Markov controls acting only on the drift coefficient. Following a martingale problem approach, he proved the existence of optimal controls under wide conditions. Nisio [125] considers the case where both the drift and diffusion coefficients are controlled. She characterizes the related value function as a viscosity solution to a nonlinear parabolic PDE of Hamilton–Jacobi–Bellman type. Successively, Claisse [42] extends the previous results by allowing controls that may not preserve the independence of the particles and considering the lifespan and the progeny coefficients to depend on the position and the control. Following the approach of Fleming and Soner [81] which relies on a result due to Krylov [106], the value function is approximated by a sequence of smooth value functions corresponding to small perturbations of the initial problem. This is what allows us to prove a Dynamic Programming Principle (DPP) and to derive a related dynamic programming equation.

This paper explores a stochastic target problem involving a controlled branching diffusion process, with branching parameters not depending on the position or the control. The problem consists of finding a minimal initial condition for a given target branching diffusion process such that it dominates a function of another controlled branching diffusion process for each particle alive at a given terminal time.

We, then, give an extended equivalent formulation of the problem. Indeed, as the starting condition of the target branching process may contain several points, the previous problem is not well-posed. Therefore, we look for the minimal value dominating all starting points such that the related branching process satisfies the terminal constraint.

Such a problem finds an application in mathematical finance when dealing with the optimal investment in crypto-currencies. For these assets, branching may appear due to their structure, leading to new assets (see, *e.g.*, [76]). In this framework, the super-replication issue remains to the best of our knowledge unsolved. Our setting provides a possible solution and we give a detailed example as an illustration.

We adopt a DPP approach to characterize the value function of our branching stochastic target problem. Contrary to [42], our argument does not rely on the existence of a regular solution to approximated PDEs but on probabilistic results. We use a measurable selection theorem similar to that of [145]. Combining it with a conditioning property for the law of the controlled process, we get the DPP.

We use it to identify the value function as a solution to a dynamic programming PDE. This is done, as in [42], using a branching property. It relates the value function at a given starting condition to the optimal values at its points. This allows us to see the value function as a sequence of functions from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ indexed by the (countable) set of particle labels. Contrary to the classical branching property, ours writes the value function as a maximum instead of a product. Hence, it entails irregularity, bringing us out of the range of regular solutions.

We, therefore, adopt the framework of viscosity solutions. The dependence on the label variable leads to adapting the definition of viscosity solutions and imposing a continuous bound in the label. Using the DPP, the value function is shown to be a viscosity solution to a partial differential inequality of two terms. The first one is the classical nonlinear second-order operator for classical diffusion processes, written as a supremum of a linear operator over controls that kill the diffusive part. The restriction to these controls is due to the terminal constraint, imposed with probability one (see, *e.g*, [26]). The second term expresses monotonicity with respect to the label. More precisely, the value function taken at some label must be greater than its value on any other offspring label.

We notice that our PDE does not contain any polynomial of the value function. This differs from classical PDEs related to branching processes as we do not work with a multiplicative cost. Moreover, the branching parameters do not impact the structure of our PDE except for the maximal size of the offspring. This is due to the specific structure of the considered control problem. We complete this parabolic PDE property with a terminal condition. The proofs of the viscosity properties follow the original lines presented in [146] and are adapted to our framework.

To get a full characterization of our value function, we finally consider the uniqueness of the PDE. Under additional assumptions, we prove a comparison theorem using the classical approach of doubling variables combined with Ishii's lemma. This shows that the value function is the unique viscosity solution to the PDE. As a byproduct, we get the continuity of the value function on the parabolic interior of the domain.

The remainder of the paper is organized as follows. In Section 1.2, we present the branching stochastic target problem and provide an example of an application inspired by fintech. In Section 1.3, we set the dynamic programming principle. We finally show in Section 1.4 the viscosity properties of the value function and provide a uniqueness result to the related PDE.

1.2 The problem

1.2.1 Branching diffusion processes

We start with a description of the underlying controlled processes. As those processes are of branching type, we first introduce the label set following the Ulam-Harris notation.

Label set For $n \geq 1$, a multi-integer $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$ is simply denoted by $i = i_1 \ldots i_n$. For $n, m \geq 1$ and two multi-integers $i = i_1 \ldots i_n \in \mathbb{N}^n$ and $j = j_1 \ldots j_m \in \mathbb{N}^m$, we define their concatenation $ij \in \mathbb{N}^{n+m}$ by

$$ij \quad := \quad i_1 \ldots i_n j_1 \ldots j_m \,. \tag{1.2.1}$$

To describe the evolution of the particle population, we introduce the set of labels $\mathcal{I}$ defined by

$$\mathcal{I} \quad := \quad \{\varnothing\} \cup \bigcup_{n=1}^{+\infty} \mathbb{N}^n ,$$

where the label $\varnothing$ corresponds to the mother particle. We extend the concatenation (1.2.1) to the whole set $\mathcal{I}$ by

$$\varnothing i \quad = \quad i\varnothing \quad = \quad i ,$$

for all $i \in \mathcal{I}$. When the particle labelled $i = i_1 \dots i_n \in \mathbb{N}^n$ gives birth to k particles, the offspring are labelled $i0, \dots, i(k-1)$. We also define the partial ordering relation $\preceq$ (resp. $\prec$) by

$$j \preceq i \quad \Leftrightarrow \quad \exists \ell \in \mathcal{I} \ : \ i = j\ell$$
$$(\text{resp. } j \prec i \quad \Leftrightarrow \quad \exists \ell \in \mathcal{I} \setminus \{\varnothing\} \ : \ i = j\ell) ,$$

for all $i, j \in \mathcal{I}$. We introduce the distance $d_{\mathcal{I}}$ on $\mathcal{I}$ defined by

$$d_{\mathcal{I}}(i,j) := \sum_{\ell=p+1}^{n} (i_\ell + 1) + \sum_{\ell'=p+1}^{m} (j_{\ell'} + 1) ,$$

for $i = i_1 \cdots i_n \in \mathbb{N}^n, j = j_1 \cdots j_m \in \mathbb{N}^m$, with

$$p \quad = \quad \max\{\ell \geq 1 \ : \ i_\ell = j_\ell\} .$$

This distance corresponds to a weighted graph distance associated to the genealogy tree embedded in $\mathcal{I}$. It is, indeed, the length of the shortest path to get from one vertex to another one, weighted by their labelling. This metric is useful to describe the phenomenon of divergence towards infinity, as it increases with the growing of the number of children and generations increases. We next write $|i| := d_{\mathcal{I}}(i, \emptyset)$ for $i \in \mathcal{I}$.

Set of finite measures In the sequel, we shall consider finite measure on $\mathcal{I} \times \mathbb{R}^\ell$ for $\ell \geq 1$. For that, we endow the set $\mathcal{I} \times \mathbb{R}^\ell$ with the metric d defined as follows

$$d\left((i,x),(j,y)\right) \quad := \quad d_{\mathcal{I}}(i,j) + |x-y| , \qquad \text{for } i,j \in \mathcal{I} \ , x,y \in \mathbb{R}^\ell .$$

$\mathcal{I} \times \mathbb{R}^\ell$ is, then, separable and complete. We denote by $\mathcal{M}_F(\mathcal{I} \times \mathbb{R}^\ell)$ the set of finite measures on $\mathcal{I} \times \mathbb{R}^\ell$. From [101, Lemma 4.5], $\mathcal{M}_F(\mathcal{I} \times \mathbb{R}^\ell)$ endowed with the topology of the weak convergence is Polish. We recall that we say that a sequence $(\nu_n)_{n\geq 0}$ weakly converges to ν in $\mathcal{M}_F(\mathcal{I} \times \mathbb{R}^\ell)$ if $\int f d\nu_n \to \int f d\nu$ as $n \to +\infty$ for any continuous and bounded function f from $\mathcal{I} \times \mathbb{R}^\ell$ to $\mathbb{R}$. A possible metric associated with the weak topology on $\mathcal{M}_F(\mathcal{I} \times \mathbb{R}^\ell)$ is the Prokhorov metric (see, *e.g.*, [101, Lemma 4.3]). We next define the subset E_ℓ of $\mathcal{M}_F(\mathcal{I} \times \mathbb{R}^\ell)$ by

$$E_\ell \quad := \quad \left\{ \sum_{i \in V} \delta_{(i,x^i)} \ ; \ V \subseteq \mathcal{I} \ , \ V \text{ finite} \ , \ x^i \in \mathbb{R}^\ell \text{ and } i \not\prec j \text{ for } i,j \in V \right\} . \qquad (1.2.2)$$

By Proposition 1.5.9, E_ℓ is Polish as well.

Probabilistic setting We fix a deterministic terminal time $T > 0$. We want to work in a filtered probability space endowed with a family of processes $(B^i, Q^i)_{i\in\mathcal{I}}$ such that

- $(B^i_t)_{t\in[0,T]}$ is an standard Brownian motion in $\mathbb{R}^m$ for all $i \in \mathcal{I}$;
- $Q^i(\mathrm{d}t, \mathrm{d}k)$ is an Poisson random measure on $[0,T]\times\mathbb{N}$ with intensity measure $\gamma\sum_{k\geq 0} p_k\delta_k$ for all $i\in\mathcal{I}$, with $\gamma > 0$, $p_k \geq 0$ for $k \geq 0$ and $\sum_{k\geq 0} p_k = 1$, δ_k being the Dirac measure at k;
- $\{B^i, Q^j\ , i, j \in \mathcal{I}\}$ forms a family of mutually independent processes.

To this purpose, we consider the setting introduced in [42] as follows.

- Let Ω^0 be the space of continuous functions from $[0,T]$ that are $\mathbb{R}^m$-valued starting at 0. Let $\mathbb{F}^0 := (\mathcal{F}^0_t)_{t\in[0,T]}$ be the filtration generated by the canonical process $B(\omega^0) := \omega^0$, $\omega^0 \in \Omega^0$. We endow $(\Omega^0, \mathcal{F}^0_T)$ with the Wiener measure $\mathbb{P}^0$.
- Let Ω^1 be the set of integer-valued Borel measures ω^1 on $[0,T]\times\mathbb{N}$, of the form $\omega^1 = \sum_{k\geq 0}\delta_{(t_k,n_k)}$, which are locally finite. Equipped with the vague topology, this space is Polish (see, *e.g.*, [101, Theorem 4.2]) Let $\mathbb{F}^1 := (\mathcal{F}^1_t)_{t\in[0,T]}$ be the filtration generated by the canonical process $Q(\omega^1) = \omega^1$:

$$\mathcal{F}^1_t := \sigma\left(Q([0,s]\times\{k\})\ : s\in[0,t],\ k\in\mathbb{N}\right), \qquad \text{for } t\in[0,T]\ .$$

  We endow $(\Omega^1, \mathcal{F}^1_T)$ with the law $\mathbb{P}^1$ of the Poisson random measure Q with intensity $\gamma\sum_{k\geq 0} p_k\delta_k$.

Following the desired structure on $\{B^i, Q^j\ , i, j \in \mathcal{I}\}$, we define $\Omega := (\Omega^0\times\Omega^1)^{\mathcal{I}}$, $\mathcal{F} := (\mathcal{F}^0_T\otimes\mathcal{F}^1_T)^{\otimes\mathcal{I}}$, and $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]} = ((\mathcal{F}^0_t\otimes\mathcal{F}^1_t)^{\otimes\mathcal{I}})_{t\in[0,T]}$. It is clear that the probability space is Polish as countable product of Polish space. In particular, this ensures the existence or regular conditional probability distributions that we shall use in the sequel. Considering the measure $\mathbb{P} := (\mathbb{P}^0\otimes\mathbb{P}^1)^{\otimes\mathcal{I}}$, we consider $\mathbb{F}^{\mathbb{P}} := (\mathcal{F}^{\mathbb{P}}_t)_{t\in[0,T]}$ (resp. $\mathcal{F}^{\mathbb{P}}$) to be the right-continuous $\mathbb{P}$-augmentation of $\mathbb{F}$ (resp. $\mathcal{F}$). On the space $\left(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{F}^{\mathbb{P}}, \mathbb{P}\right)$, we extend the definition of the processes B^i and Q^i for $i\in\mathcal{I}$ as the previously described processes B and Q composed with the projections on each component, *i.e.*,

$$B^i(\omega) := \omega^{0,i},\ Q^j(\omega) := \omega^{1,j}, \quad \omega = (\omega^{0,i}, \omega^{1,i})_{i\in\mathcal{I}}\in\Omega\ .$$

To stress the dependence on time, we will use the following notations. For $t\in[0,T]$ and $\omega = (\omega^0, \omega^1)\in\Omega$, we define the stopped path at time t by $\omega_{.\wedge t} := (\omega^0_{.\wedge t}, \omega^1_{.\wedge t})$, where

$$\omega^0_{.\wedge t} \ = \ (\omega^0_{s\wedge t})_{s\geq 0} \quad \text{and} \quad \omega^1_{.\wedge t} \ = \ \omega^1(\cdot\cap[0,t]\times\mathbb{N})\ .$$

For a process $(X_t)_{t\in[0,T]}$ and a random time $\tau:\ \Omega\to[0,T]$, we denote by $(X_{t\wedge\tau})_{t\in[0,T]}$ the process X defined by

$$X_{t\wedge\tau}(\omega) \ := \ X_{t\wedge\tau(\omega)}(\omega)\ , \quad \text{for } t\in[0,T],\ \omega\in\Omega\ .$$

For $\omega,\ \tilde\omega\in\Omega$ and a random time $\tau:\ \Omega\to[0,T]$, we define the concatenation path $\omega\oplus_\tau\tilde\omega = (\omega^{0,i}\oplus_\tau\tilde\omega^{0,i}, \omega^{1,i}\oplus_\tau\tilde\omega^{1,i})_{i\in\mathcal{I}}$ by

$$(\omega^{0,i}\oplus_\tau\tilde\omega^{0,i})_s \ := \ \omega^{0,i}_s\mathbb{1}_{s<\tau(\omega)} + (\tilde\omega^{0,i}_s - \tilde\omega^{0,i}_{\tau(\omega)} + \omega^{0,i}_{\tau(\omega)})\mathbb{1}_{s\geq\tau(\omega)}\ , \quad \text{for } s\in[0,T]\ ,$$

and

$$\omega^{1,i} \oplus_\tau \tilde{\omega}^{1,i} \quad = \quad \omega^{1,i}(\cdot \cap [0,\tau(\omega)] \times \mathbb{N}) + \tilde{\omega}^{1,i}(\cdot \cap (\tau(\omega),T] \times \mathbb{N}) \ ,$$

for $i \in \mathcal{I}$. For a random variable S valued in some Polish space, we also define the shifted random variable $S^{\tau,\omega}$ by

$$S^{\tau,\omega}(\tilde{\omega}) \quad := \quad S(\omega \oplus_\tau \tilde{\omega}) \ , \quad \tilde{\omega} \in \Omega \ . \tag{1.2.3}$$

Alive particles We denote by $\mathcal{V}_t$ the set of alive particles at time t. It is a finite set that changes only at times of an increasing sequence of stopping times. In particular, $\mathcal{V}_{t-}$ is well defined $\mathbb{P}$-a.s. as the value before a possible change at t. More precisely we define $\mathcal{V}$ as follows.

- At time $t = 0$, the set is reduced to the mother particle : $\mathcal{V}_0 := \{\varnothing\}$.
- For a time $t \geq 0$, given $\{i \in \mathcal{V}_t\}$, the particle i dies at the first time τ_i the related Poisson measure Q^i jumps after t, *i.e.*,

$$\tau_i \quad = \quad \inf\{s > t \ : \ Q^i((t,s] \times \mathbb{N}) = 1\} \ .$$

- At time τ_i, this particle gives birth to k particles $i0, \ldots, i(k-1)$, with k such that $Q^i(\{\tau_i\} \times \{k\}) = 1$:

$$\mathcal{V}_{\tau_i} \quad = \quad (\mathcal{V}_{\tau_i -} \setminus \{i\}) \cup \{i0, \ldots, i(k-1)\} \ .$$

Controlled population Take A a Polish space with metric d_A. We assume d_A to be bounded (if not so, we replace d_A with $d_A \wedge 1$ and still have a Polish space).

Definition 1.2.1. *We say that $\alpha = (\alpha^i)_{i \in \mathcal{I}}$ is an* admissible control *if α^i is $\mathbb{F}$-predictable process valued in A, for any $i \in \mathcal{I}$. We denote by $\mathcal{A}$ the set of such controls.*

Let $\lambda : \ \mathbb{R}^d \times A \to \mathbb{R}^d$ and $\sigma : \ \mathbb{R}^d \times A \to \mathbb{R}^{d \times m}$ be measurable functions. For a given control $\alpha \in \mathcal{A}$, each particle $i \in \mathcal{I}$ of the controlled population is born, evolves, and dies to give birth to offspring according to the set $\mathcal{V}$ defined above. We denote by X^i_s the position at time s of a particle $i \in \mathcal{V}_s$. For $i \in \mathcal{I}$ alive at time t, let $\tau_i \geq t$ be its death time, giving birth to k offspring. The position at a time $s \geq \tau_i$ of the offspring $i0, \ldots, i(k-1)$ are given by

$$X^{i\ell}_{\tau_i} \quad = \quad X^i_{\tau_i -} \ , \tag{1.2.4}$$
$$\mathrm{d}X^{i\ell}_s \quad = \quad \lambda(X^{i\ell}_s, \alpha^{i\ell}_s)\mathrm{d}s + \sigma(X^{i\ell}_s, \alpha^{i\ell}_s)\mathrm{d}B^{i\ell}_s \ , \tag{1.2.5}$$

for $\ell = 0, \ldots, k-1$, such that $i\ell$ is alive at time s. We represent the population of alive particles by the following measure-valued process

$$Z_s \quad = \quad \sum_{i \in \mathcal{V}_s} \delta_{(i, X^i_s)} \ , \qquad \text{for } s \geq 0 \ .$$

The process Z takes values in the Polish space E_d defined by (1.2.2).

For a function $f = (f_i)_{i \in \mathcal{I}}$ such that $f_i : \mathbb{R}^d \to \mathbb{R}$, and a measure $\mu = \sum_{i \in V} \delta_{(i,x_i)} \in E_d$, we set

$$f(\mu) \quad := \quad \int_{\mathcal{I} \times \mathbb{R}^d} f d\mu \quad = \quad \sum_{i \in V} f_i(x_i) \ .$$

We introduce the second order local operators L^a, $a \in A$ defined by

$$L^a\varphi(x) \quad := \quad \lambda(x,a)^\top D\varphi(x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,a)D^2\varphi(x)\right) \ , \qquad \text{for } x \in \mathbb{R}^d \ ,$$

for $\varphi \in C^2(\mathbb{R}^d)$, where $D\varphi$ and $D^2\varphi$ denote respectively, the gradient and the Hessian matrix of φ.

For a control $\alpha \in \mathcal{A}$ and a function $f = (f_i)_{i\in\mathcal{I}}$ such that $f_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $f_i \in C^{1,2}([0,T] \times \mathbb{R}^d)$, for $i \in \mathcal{I}$, the following SDE characterises the behaviour of Z:

$$\begin{aligned} f(t,Z_t) \quad = \quad & f(s,Z_s) + \int_s^t \sum_{i\in\mathcal{V}_u} Df_i(u,X_u^i)^\top \sigma(X_u^i,\alpha_u^i)\mathrm{d}B_u^i \\ & + \int_s^t \sum_{i\in\mathcal{V}_u} (\partial_t + L^{\alpha_u^i})f_i(u,X_u^i)\mathrm{d}u + \int_{(s,t]} \sum_{i\in\mathcal{V}_{u-}} \sum_{k\geq 0} \left(\sum_{\ell=0}^{k-1} f_{i\ell} - f_i\right)(u,X_u^i)Q^i(\mathrm{d}u\mathrm{d}k) \ , \end{aligned} \tag{1.2.6}$$

for all $s,t \in [0,T]$ such that $s \leq t$.

Target branching diffusion process To each alive particle $i \in \mathcal{V}_s$, we associate a target position at time s denoted by Y_s^i. Let $\lambda_Y : \mathbb{R}^d \times \mathbb{R} \times A \to \mathbb{R}$ and $\sigma_Y : \mathbb{R}^d \times \mathbb{R} \times A \to \mathbb{R}^{1\times m}$ be measurable functions. Let $\tau_i \geq t$ be the death time of $i \in \mathcal{I}$. Conditionally to the event $\{s \geq \tau_i\}$, the target position of any child of the particle i is given by

$$Y_{\tau_i}^{i\ell} \quad = \quad Y_{\tau_i -}^i \ , \tag{1.2.7}$$

$$dY_s^{i\ell} \quad = \quad \lambda_Y(X_s^{i\ell},Y_s^{i\ell},\alpha_s^{i\ell})\,\mathrm{d}s + \sigma_Y(X_s^{i\ell},Y_s^{i\ell},\alpha_s^{i\ell})\,\mathrm{d}B_s^{i\ell} \ , \tag{1.2.8}$$

such that particle $i\ell$ is alive at time s.

We use the notation $\hat{\cdot}$ to define the quantities associated with the pair $\binom{X_s^i}{Y_s^i}$, considering the previous problem on $\mathbb{R}^{d+1}$. Therefore, we have $\hat{X}_s^i := \binom{X_s^i}{Y_s^i}$, $\hat{\lambda}(\hat{X}_s^i,\alpha_s^i) := \binom{\lambda(X_s^i,\alpha_s^i)}{\lambda_Y(X_s^i,Y_s^i,\alpha_s)}$ and $\hat{\sigma}(\hat{X}_s^i,\alpha_s^i) := \binom{\sigma(X_s^i,\alpha_s^i)}{\sigma_Y(X_s^i,Y_s^i,\alpha_s)}$. Under those hypotheses, assuming i is alive, its position $\hat{X}^i$ evolves according to the following SDE

$$d\hat{X}_s^i \quad = \quad \hat{\lambda}(\hat{X}_s^i,\alpha_s^i)\mathrm{d}s + \hat{\sigma}(\hat{X}_s^i,\alpha_s^i)\,\mathrm{d}B_s^i \ . \tag{1.2.9}$$

The resulting population process valued in E_{d+1} is

$$\hat{Z}_s \quad = \quad \sum_{i\in\mathcal{V}_s} \delta_{(i,X_s^i,Y_s^i)} \ , \qquad \text{for } s \geq 0 \ .$$

As before, we define the related second-order local operators $\hat{L}^a$, $a \in A$ by

$$\hat{L}^a\hat{\varphi}(\hat{x}) \quad = \quad \hat{\lambda}(\hat{x},a)^\top D\hat{\varphi}(\hat{x}) + \frac{1}{2}\mathrm{Tr}(\hat{\sigma}\hat{\sigma}^\top(\hat{x},a)D^2\hat{\varphi}(\hat{x})) \ , \qquad \text{for } \hat{x} \in \mathbb{R}^{d+1} \ ,$$

for $\hat{\varphi} \in C^2(\mathbb{R}^{d+1})$.

For a control $\alpha \in \mathcal{A}$ and a function $\hat{f} :\ [0,T] \times \mathcal{I} \times \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\hat{f}_i(\cdot) \in C^{1,2}([0,T] \times$

$\mathbb{R}^{d+1}$) for all $i \in \mathcal{I}$, the SDE related to $\hat{Z}$ takes the following form:

$$\begin{aligned} \hat{f}(t, \hat{Z}_t) &= \hat{f}(s, \hat{Z}_s) + \int_s^t \sum_{i \in \mathcal{V}_u} D\hat{f}_i(u, \hat{X}_u^i)^\top \hat{\sigma}(\hat{X}_u^i, \alpha_u^i) \mathrm{d}B_u^i \\ &\quad + \int_s^t \sum_{i \in \mathcal{V}_u} (\partial_t + \hat{L}^{\alpha_u^i}) \hat{f}_i(u, \hat{X}_u^i) \mathrm{d}u + \int_{(s,t]} \sum_{i \in \mathcal{V}_{u-}} \sum_{k \geq 0} \left(\sum_{\ell=0}^{k-1} \hat{f}_{i\ell} - \hat{f}_i \right) (u, \hat{X}_u^i) Q^i(\mathrm{d}u\mathrm{d}k) , \end{aligned} \tag{1.2.10}$$

for all $s, t \in [0, T]$ such that $s \leq t$.

We make the following assumption to ensure the well-posedness of the presented controlled processes.

Assumption A1. *(i) The coefficients p_k, $k \geq 0$, satisfy*

$$\sum_{k \geq 0} k p_k = M < +\infty .$$

(ii) The functions λ, σ, λ_Y and σ_Y satisfy

$$\sup_{a \in A} |\lambda(0, a)| + |\sigma(0, a)| + |\lambda_Y(0, 0, a)| + |\sigma_Y(0, 0, a)| < +\infty .$$

(iii) There exists a constant $L > 0$ such that

$$\begin{aligned} |\lambda(x, a) - \lambda(x', a)| + |\sigma(x, a) - \sigma(x', a)| + |\lambda_Y(x, y, a) - \lambda_Y(x', y', a)| \\ + |\sigma_Y(x, y, a) - \sigma_Y(x', y', a)| \leq L \left(|x - x'| + |y - y'| \right) , \end{aligned}$$

for all $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$ and $a \in A$.

(iv) There exists a nondecreasing bounded function $w : \mathbb{R}_+ \to \mathbb{R}_+$ such that $w(r) \xrightarrow[r \to 0]{} 0$ and

$$\begin{aligned} |\lambda(x, a) - \lambda(x, a')| + |\sigma(x, a) - \sigma(x, a')| + |\lambda_Y(x, y, a) - \lambda_Y(x, y, a')| \\ + |\sigma_Y(x, y, a) - \sigma_Y(x, y, a')| \leq w \left(d_A(a, a') \right) , \end{aligned}$$

for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $a, a' \in A$.

For any initial condition $t \in [0, T]$, $\mu = \sum_{i \in V} \delta_{(i, x_i)} \in E_d$ and $y_i \in \mathbb{R}$ for $i \in V$, we extend the controlled branching processes (X, Y). For that, the set of alive particles $\mathcal{V}^{t,\mu}$ is defined as follows.

- For $s \in [0, t]$, $\mathcal{V}_s^{t,\mu} := V$.
- For $s \geq t$, a particle $i \in \mathcal{V}_s$ dies at the first time τ_i the related Poisson measure Q^i jumps after s:

$$\tau_i = \inf\{r > s \ : \ Q^i(]s, r] \times \mathbb{N}) = 1\} .$$

- At time τ_i, the particle i gives birth to k particles $i0, \ldots, i(k-1)$, with k such that $Q^i(\{\tau_i\} \times \{k\}) = 1$:

$$\mathcal{V}_{\tau_i}^{t,\mu} := \left(\mathcal{V}_{\tau_i -}^{t,\mu} \setminus \{i\} \right) \cup \{i0, \ldots, i(k-1)\} .$$

Then, the controlled branching population process $X^{t,\mu,\alpha} = (X_s^{t,\mu,\alpha,i}, i \in \mathcal{V}_s^{t,\mu})_{s\in[0,T]}$ is defined by the initial condition

$$X_s^{t,\mu,\alpha} \quad := \quad (x_i,\ i \in V)\ , \quad s \in [0,t]\ ,$$

together with dynamics (1.2.4)-(1.2.5). We also denote by $\hat{\mu} \in E_{d+1}$ the extended measure as

$$\hat{\mu} \quad := \quad \sum_{i\in V} \delta_{(i,x_i,y_i)}\ ,$$

and $Y^{t,\hat{\mu},\alpha} = (Y_s^{t,\hat{\mu},\alpha,i}, i \in \mathcal{V}_s^{t,\mu})_{s\in[0,T]}$ the controlled branching target process with initial condition

$$Y_s^{t,\hat{\mu},\alpha,i} \quad := \quad y_i\ , \quad s \in [0,t]\ ,$$

for all $i \in V$, together with dynamics (1.2.7)-(1.2.8). Let $Z^{t,\mu,\alpha}$ and $\hat{Z}^{t,\hat{\mu},\alpha}$ be

$$Z_s^{t,\mu,\alpha} := \sum_{i\in\mathcal{V}_s^{t,\mu}} \delta_{(i,X_s^{t,\mu,\alpha,i})} \quad \text{and} \quad \hat{Z}_s^{t,\hat{\mu},\alpha} := \sum_{i\in\mathcal{V}_s^{t,\mu}} \delta_{(i,X_s^{t,\mu,\alpha,i},Y_s^{t,\hat{\mu},\alpha,i})}\ ,$$

for $s \in [0,T]$.

In this setting, we have the following non-explosion result.

Proposition 1.2.3. *Suppose that Assumptions A1 (i)-(ii)-(iii) hold. Fix* $t \in [0,T]$, $\mu = \sum_{i\in V} \delta_{(i,x_i)} \in E_d$, $\hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y_i)} \in E_{d+1}$ *and* $\alpha \in \mathcal{A}$.

(i) The set of alive particles $\mathcal{V}_s^{t,\mu}$ *is uniquely defined and is finite for all* $s \in [0,T]$*. More precisely, we have*

$$\mathbb{E}\left[\sup_{s\in[0,T]} |\mathcal{V}_s^{t,\mu}|\right] \quad \leq \quad |V|e^{\gamma M(T-t)}\ ,$$

where $|\mathcal{V}|$ *stands for the cardinal of a subset* $\mathcal{V}$ *of* $\mathcal{I}$.

(ii) There exists a unique $\mathbb{F}^{\mathbb{P}}$*-adapted process* $(Z^{t,\mu,\alpha})$ *(resp.* $(\hat{Z}^{t,\hat{\mu},\alpha})$*) valued in* E_d *(resp.* E_{d+1}*). Moreover, the process* $Z^{t,\mu,\alpha}$ *(resp.* $(\hat{Z}^{t,\hat{\mu},\alpha})$*) satisfies* (1.2.6) *(resp.* (1.2.10)*).*

We refer to [42, Proposition 2.1] for the proof of this proposition.

Remark 1.2.1. *For any* $i \in \mathcal{I}$*, the processes* $X^{t,\mu,\alpha,i}$ *and* $Y^{t,\hat{\mu},\alpha,i}$ *are defined on times* $s \in [t,T]$ *such that* $i \in \mathcal{V}_s^{t,\mu}$*. However, we can extend their definition to the whole interval* $[t,T]$ *and to any* $i \in \mathcal{I}$ *such that* $j \preceq i$ *for* $j \in \mathcal{V}_t^{t,\mu}$*. In this case, there exists* $k \geq 1$ *and* $\ell_1, \ldots, \ell_k$ *such that*

$$i \quad = \quad j\ell_1 \ldots \ell_k\ .$$

We denote the associated branching times by $(S_0, \ldots, S_k)$ *defined as follows*

$$S_m \quad = \quad \inf\left\{s > S_{m-1}\ : \exists n \geq \ell_{m+1}+1, Q^{j\ell_1\ldots\ell_m}((S_{m-1},s]\times\{n\}) = 1\right\}\ ,$$

for $m = 0, \ldots, k-1$ *with* $S_{-1} = t$*. Then, we define the extended processes* $X^{t,\mu,\alpha,i}$ *and* $Y^{t,\mu,\alpha,i}$

by

$$\begin{aligned} X_s^{t,\mu,\alpha,i} &:= \mathbb{1}_{[t,S_0)}(s)X_s^{t,\mu,\alpha,j} + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)X_s^{t,\mu,\alpha,j\ell_1\ldots\ell_m} + \mathbb{1}_{[S_{k-1},+\infty)}(s)X_s^{t,\mu,\alpha,i} , \\ Y_s^{t,\hat{\mu},\alpha,i} &:= \mathbb{1}_{[t,S_0)}(s)Y_s^{t,\hat{\mu},\alpha,j} + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)Y_s^{t,\hat{\mu},\alpha,j\ell_1\ldots\ell_m} + \mathbb{1}_{[S_{k-1},+\infty)}(s)Y_s^{t,\hat{\mu},\alpha,i} , \end{aligned}$$

for $s \in [t,T]$.

These extended processes can be seen as the solution to a Brownian stochastic differential equation with Lipschitz coefficients. Obvious in the first case, to show it in the second one, we consider the ancestor Brownian motion $\bar{B}^i$ *defined by*

$$\begin{aligned} \bar{B}_s^i &:= B_s^j\mathbb{1}_{[t,S_0)} + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s) \left(B_s^{j\ell_1\ldots\ell_m} - B_{S_{m-1}}^{j\ell_1\ldots\ell_m} + B_{S_{m-1}}^{j\ell_1\ldots\ell_{m-1}} \right) \\ &\qquad + \mathbb{1}_{[S_{k-1},+\infty)}(s) \left(B_s^i - B_{S_{k-1}}^i + B_{S_{k-1}}^{j\ell_1\ldots\ell_{k-1}} \right) , \end{aligned}$$

for $s \in [t,T]$. *This process is continuous, centered, with independent increments and variance equal to t, therefore, a Brownian motion by Lévy's characterization. Then, the extended processes* $X^{t,\mu,\alpha,i}$ *and* $Y^{t,\mu,\alpha,i}$ *are the unique solutions to the SDE*

$$dX_s^{t,\mu,\alpha,i} = \bar{\lambda}(s, X_s^{t,\mu,\alpha,i})ds + \bar{\sigma}(s, X_s^{t,\mu,\alpha,i})d\bar{B}_s^i , \tag{1.2.11}$$

$$dY_s^{t,\hat{\mu},\alpha,i} = \bar{\lambda}_Y(s, X_s^{t,\mu,\alpha,i}, Y_s^{t,\hat{\mu},\alpha,i})\, ds + \bar{\sigma}_Y(s, X_s^{t,\mu,\alpha,i}, Y_s^{t,\hat{\mu},\alpha,i}))\, d\bar{B}_s^i , \tag{1.2.12}$$

for $s \in [t,T]$, *with initial condition* $X_t^{t,\mu,\alpha,i} = x_i$ *and* $Y_t^{t,\mu,\alpha,i} = y_i$. *The coefficients are given by*

$$\begin{aligned} \bar{\lambda}(s,x) &= \mathbb{1}_{[t,S_0)}\lambda(x,\alpha_s^j) + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)\lambda(x,\alpha_s^{j\ell_1\ldots\ell_m}) + \mathbb{1}_{[S_{k-1},+\infty)}(s)\lambda(x,\alpha_s^i) , \\ \bar{\sigma}(s,x) &= \mathbb{1}_{[t,S_0)}\sigma(x,\alpha_s^j) + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)\sigma(x,\alpha_s^{j\ell_1\ldots\ell_m}) + \mathbb{1}_{[S_{k-1},+\infty)}(s)\sigma(x,\alpha_s^i) , \\ \bar{\lambda}_Y(s,x,y) &= \mathbb{1}_{[t,S_0)}\lambda_Y(x,y,\alpha_s^j) + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)\lambda_Y(x,y,\alpha_s^{j\ell_1\ldots\ell_m}) + \mathbb{1}_{[S_{k-1},+\infty)}(s)\lambda_Y(x,y,\alpha_s^i) , \\ \bar{\sigma}_Y(s,x,y) &= \mathbb{1}_{[t,S_0)}\sigma_Y(x,y,\alpha_s^j) + \sum_{m=1}^{k-1} \mathbb{1}_{[S_{m-1},S_m)}(s)\sigma_Y(x,y,\alpha_s^{j\ell_1\ldots\ell_m}) + \mathbb{1}_{[S_{k-1},+\infty)}(s)\sigma_Y(x,y,\alpha_s^i) , \end{aligned}$$

for $(s,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$. *Under Assumption A1, these coefficients satisfy classical Lipschitz and boundedness assumption to have uniqueness and stability of solutions. In the sequel, we shall refer by* $X^{t,\mu,\alpha,i}$ *and* $Y^{t,\hat{\mu},\alpha,i}$ *either to the processes themselves or to their extended definitions if the processes are considered outside their living interval. We also remark that we can build* $X^{t,\mu,\alpha,i}$ *(resp.* $Y^{t,\hat{\mu},\alpha,i}$*) is built as the solution to* (1.2.4)-(1.2.5) *(resp.* (1.2.7)-(1.2.8)*) even when there is no* $j \notin \mathcal{V}_t^{t,\mu}$ *such that* $j \preceq i$.

Under the additional regularity assumption on the coefficients with respect to the control, we have a stability result for the branching system.

Proposition 1.2.4. *Suppose that Assumption A1 holds and fix* $t \in [0,T]$, $\mu = \sum_{i \in V} \delta_{(i,x_i)} \in E_d$,

$\hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y_i)} \in E_{d+1}$, and $\alpha \in \mathcal{A}$. Let $(t_n)_{n\geq 1}$, $\big(\hat{\mu}_n = \sum_{i\in V_n} \delta_{(i,x_i^n,y_i^n)}\big)_{n\geq 1}$, and $(\alpha^n)_{n\geq 1}$ be sequences of $\mathbb{R}_+$, E_{d+1}, and $\mathcal{A}$ such that

$$(t_n, \hat{\mu}_n) \xrightarrow[n\to+\infty]{} (t, \hat{\mu}) \ , \tag{1.2.13}$$

and

$$\mathbb{E}\int_0^T d_A\left(\alpha_s^i, \alpha_s^{n,i}\right) ds \xrightarrow[n\to+\infty]{} 0 \ , \tag{1.2.14}$$

for all $i \in \mathcal{I}$. Then,

$$\mathbb{E}\left[|X_s^{t_n,\mu_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - X_s^{t,\mu,\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2 + |Y_s^{t_n,\hat{\mu}_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - Y_s^{t,\hat{\mu},\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2\right] \xrightarrow[n\to+\infty]{} 0 \ ,$$

for all $s \in [t,T]$ and $i \in \mathcal{I}$, where $\mu_n := \sum_{i\in V_n} \delta_{(i,x_i^n)} \in E_d$ for any $n \geq 1$.

Proof. From (1.2.13), we have

$$V_n = V \ , \tag{1.2.15}$$

for n large enough. We take such an n in the following. Consider the event $\mathcal{E}_n$ defined as follows

$$\mathcal{E}_n := \bigcap_{i\in V} \left\{Q^i\left([t_n\wedge t, t_n\vee t]\times\mathbb{N}\right) = 0\right\} .$$

Since $t_n \to t$ as $n \to \infty$, we have that $\lim_{n\to\infty}\mathbb{P}(\mathcal{E}_n^c) = 0$. Moreover, combining the extension for the processes X^i and Y^i discussed in Remark 1.2.1 with standard results on the second order moment for SDE, we have that

$$\begin{aligned}\mathbb{E}&\left[|X_s^{t_n,\mu_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - X_s^{t,\mu,\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2 + |Y_s^{t_n,\hat{\mu}_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - Y_s^{t,\hat{\mu},\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2\right]\\ &\leq 4E\left[|X_s^{t_n,\mu_n,\alpha_n,i}|^2 + |X_s^{t,\mu,\alpha,i}|^2 + |Y_s^{t_n,\hat{\mu}_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}}|^2 + |Y_s^{t,\hat{\mu},\alpha,i}|^2\right] < +\infty \ .\end{aligned}$$

This means that, from dominated convergence theorem, we obtain

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{E}_n^c}\left(|X_s^{t_n,\mu_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - X_s^{t,\mu,\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2 + |Y_s^{t_n,\hat{\mu}_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - Y_s^{t,\hat{\mu},\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}|^2\right)\right] \xrightarrow[n\to+\infty]{} 0 \ .$$

We notice that the event

$$\left\{\mathcal{V}_s^{t_n,\mu_n} = \mathcal{V}_s^{t,\mu}, \quad \text{for } s\in[t_n\wedge t, T]\right\}$$

is included in $\mathcal{E}_n$, as $V_n = V$ and the same Poisson processes generate the genealogy. We, therefore, have

$$\begin{aligned}\mathbb{E}&\left[\mathbb{1}_{\mathcal{E}_n}\left(\left|X_s^{t_n,\mu_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - X_s^{t,\mu,\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}\right|^2 + \left|Y_s^{t_n,\hat{\mu}_n,\alpha_n,i}\mathbb{1}_{i\in\mathcal{V}_s^{t_n,\mu_n}} - Y_s^{t,\hat{\mu},\alpha,i}\mathbb{1}_{i\in\mathcal{V}_s^{t,\mu}}\right|^2\right)\right] \leq\\ &\mathbb{E}\left[\left|X_s^{t_n,\mu_n,\alpha_n,i} - X_s^{t,\mu,\alpha,i}\right|^2 + \left|Y_s^{t_n,\hat{\mu}_n,\alpha_n,i} - Y_s^{t,\hat{\mu},\alpha,i}\right|^2\middle|\,\mathcal{E}_n\right]\mathbb{P}\left(\mathcal{E}_n\right) \ .\end{aligned}$$

Since the processes B^j, $j \in \mathcal{I}$ are still independent Brownian motions given $\mathcal{E}_n$, we get, from

Assumption A1 (iv) and [105, Theorem 2.5.9], that there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\left|X_s^{t_n,\mu_n,\alpha_n,i} - X_s^{t,\mu,\alpha,i}\right|^2 + \left|Y_s^{t_n,\hat{\mu}_n,\alpha_n,i} - Y_s^{t,\hat{\mu},\alpha,i}\right|^2 \middle| \mathcal{E}_n\right] \leq \tag{1.2.16}$$

$$C\Bigg(\mathbb{E}\left[\sup_{u\in[t\wedge t_n, t\vee t_n]}\left|X_u^{t_n,\mu_n,\alpha_n,i} - X_u^{t,\mu,\alpha,i}\right|^2 + \left|Y_u^{t_n,\hat{\mu}_n,\alpha_n,i} - Y_u^{t,\hat{\mu},\alpha,i}\right|^2 \middle| \mathcal{E}_n\right]$$
$$+\sum_{j\preceq i}\int_{t\wedge t_n}^T \mathbb{E}\left[w\left(d\left(\alpha_s^j, \alpha_s^{n,j}\right)\right)^2 \middle| \mathcal{E}_n\right]\Bigg).$$

Condition (1.2.13) leads to $X_{t_n}^{t_n,\mu_n,\alpha_n,i} \to X_t^{t,\mu,\alpha,i}$ and $Y_{t_n}^{t_n,\hat{\mu}_n,\alpha_n,i} \to Y_t^{t,\hat{\mu},\alpha,i}$ as $n \to +\infty$. From [105, Corollary 2.5.10], we get

$$\mathbb{E}\left[\sup_{u\in[t\wedge t_n, t\vee t_n]}\left|X_u^{t_n,\mu_n,\alpha_n,i} - X_u^{t,\mu,\alpha,i}\right|^2 + \left|Y_u^{t_n,\hat{\mu}_n,\alpha_n,i} - Y_u^{t,\hat{\mu},\alpha,i}\right|^2 \middle| \mathcal{E}_n\right] \xrightarrow[n\to+\infty]{} 0 \tag{1.2.17}$$

Moreover, using condition (1.2.14), the definition of w in Assumption A1 (iv) and the fact that $\mathbb{P}(\mathcal{E}_n) \to 1$ as $n \to 0$, we obtain

$$\sum_{j\preceq i}\int_{t\wedge t_n}^T \mathbb{E}\Big[w\big(d\big(\alpha_s^j, \alpha_s^{n,j}\big)\big)^2 \Big| \mathcal{E}_n\Big] \xrightarrow[n\to+\infty]{} 0\,. \tag{1.2.18}$$

Finally, combining (1.2.16), (1.2.17) and (1.2.18), we get

$$\mathbb{E}\left[\left|X_s^{t_n,\mu_n,\alpha_n,i} - X_s^{t,\mu,\alpha,i}\right|^2 + \left|Y_s^{t_n,\hat{\mu}_n,\alpha_n,i} - Y_s^{t,\hat{\mu},\alpha,i}\right|^2 \middle| \mathcal{E}_n\right] \xrightarrow[n\to+\infty]{} 0\,,$$

which entails the result.

□

Focusing on conditional laws of the controlled processes, we use the representation result close to [43, Proposition 3.4] for admissible controls. In particular, we see that, following this proof, a process $\alpha = (\alpha^i)_{i\in\mathcal{I}}$ is an admissible control if and only if, for every $i \in \mathcal{I}$, there exists a function $\tilde{\alpha}^i : [0,T]\times\Omega \to A$ predictable w.r.t. $\mathbb{F}$ such that

$$\alpha_s^i(\omega) = \tilde{\alpha}^i\left(s, \left(B^j(\omega), Q^j(\omega)\right)_{j\in\mathcal{I}}\right) = \tilde{\alpha}^i\left(s, \left(B^j_{s\wedge\cdot}(\omega), Q^j|_{[0,s)}(\omega)\right)_{j\in\mathcal{I}}\right).$$

For $t \in [0,T]$, we denote by $\mathcal{T}_{[t,T]}$ the set of $\mathbb{F}^{\mathbb{P}}$-stopping times valued in $[t,T]$. For $\alpha \in \mathcal{A}$, $\tau \in \mathcal{T}_{[t,T]}$ and $\omega \in \Omega$, we define the control $\alpha^{\tau(\omega),\omega}$ by

$$\left(\tilde{\alpha}^{\tau(\omega),\omega}\right)^i\left(s, \left(B^j(\tilde{\omega}), Q^j(\tilde{\omega})\right)_{j\in\mathcal{I}}\right) := \tilde{\alpha}^i\left(\cdot, \left(B^j_{\tau(\omega)\wedge\cdot}\left(\omega\oplus_\tau\tilde{\omega}\right), Q^j|_{[0,\omega)}\left(\omega\oplus_\tau\tilde{\omega}\right)\right)_{j\in\mathcal{I}}\right),$$

for $i \in \mathcal{I}$, $s \geq 0$ and $\tilde{\omega} \in \Omega$. We remark that this property stands for general probability spaces.

We can now examine the pseudo-Markov property (also known as the conditioning property) for controlled branching diffusion processes. This crucial result will be used to establish the dynamic programming principle and is a consequence of the independence of increments of the processes that generate the considered filtrations.

Theorem 1.2.2 (Pseudo-Markov property)**.** *Suppose that Assumption A1 holds and fix* $(t, \hat{\mu}, \alpha) \in [0, T] \times E_{d+1} \times \mathcal{A}$ *and* $\tau \in \mathcal{T}_{[t,T]}$*. Then, for any bounded measurable function* $f : \mathbb{D}([0, T], E_{d+1}) \to \mathbb{R}$*, we have*

$$\mathbb{E}\left[f\left(\hat{Z}^{t,\hat{\mu},\alpha}\right)\Big|\mathcal{F}^{\mathbb{P}}_{\tau}\right](\omega) \quad = \quad F\left(\tau(\omega), \hat{Z}^{t,\hat{\mu},\alpha}_{.\wedge\tau}(\omega), \alpha^{\tau(\omega),\omega}\right) , \qquad \mathbb{P}(d\omega) - a.s. ,$$

where

$$F(s, \hat{z}, \beta) \quad := \quad \mathbb{E}\left[f\left(\left(\hat{z}_u \mathbb{1}_{u<s} + \hat{Z}^{s,\hat{z}_s,\beta}_u \mathbb{1}_{u\geq s}\right)_{u\in[0,T]}\right)\right] ,$$

for all $s \in [0, T]$*,* $\hat{z} \in \mathbb{D}([0, T], E_{d+1})$*, and* $\beta \in \mathcal{A}$*.*

The proof of this conditioning property follows the same lines to that in [43, Lemma 3.7], wherein the author extends the results presented in [45, Theorem 2.2]. The primary distinction in our context is the selection of $\mathbb{F}$-predictable controls α^i for $i \in \mathcal{I}$. Nevertheless, this modification does not significantly alter the proof of the conditioning property, as the same reasoning and arguments apply. For detailed insights, we refer the reader to [43].

1.2.2 The stochastic target problem

To define the stochastic target problem, let $g : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying the following assumption.

Assumption A2. *The function* g_i *is continuous on* $\mathbb{R}^d$ *for all* $i \in \mathcal{I}$*.*

Fix an initial time $t \in [0, T]$ and an initial population $\mu = \sum_{i\in V} \delta_{(i,x_i)}$. We look for an initial position y for the target process and a control $\alpha \in \mathcal{A}$ such that

$$Y^{t,\hat{\mu},\alpha,i}_t \quad = \quad y , \qquad \text{for } i \in V ,$$

and $Y^{t,\hat{\mu},\alpha}$ and $X^{t,\mu,\alpha}$ satisfy the terminal constraints

$$Y^{t,\hat{\mu},\alpha,i}_T \quad \geq \quad g_i(X^{t,\mu,\alpha,i}_T) , \qquad \text{for } i \in \mathcal{V}^{t,\mu}_T .$$

More precisely, we look for the *reachability set*

$$\mathcal{R}(t, \mu) \quad = \quad \Big\{y \in \mathbb{R}, \ : \ \exists \alpha \in \mathcal{A} \ : \ Y^{t,\hat{\mu},\alpha,i}_T \ \geq \ g_i(X^{t,\mu,\alpha,i}_T) , \ \text{ for } i \in \mathcal{V}^{t,\mu}_T \qquad \text{with } \hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y)}\Big\} , \tag{1.2.19}$$

for $t \in [0, T]$ and $\mu = \sum_{i\in V} \delta_{(i,x_i)} \in E_d$. We remark that the reachability set satisfies the following *monotonicity property.*

Proposition 1.2.5. *Suppose that Assumption A1 holds. For* $\mu = \sum_{i\in V} \delta_{(i,x_i)} \in E_d$ *and* $y \in \mathcal{R}(t, \mu)$ *we have* $[y, \infty[\subseteq \mathcal{R}(t, \mu)$*.*

Proof. Fix a control $\alpha = (\alpha^i)_{i\in\mathcal{I}}$ and a starting point (t, μ). We take $y \in \mathcal{R}(t, \mu)$, $y' \geq y$ and write $\hat{\mu}$ (resp. $\hat{\mu}'$) for $\sum_{i\in V} \delta_{(i,x_i,y)}$ (resp. $\sum_{i\in V} \delta_{(i,x_i,y')}$), Y^i (resp. Y'^i) for $Y^{t,\hat{\mu},\alpha,i}$ (resp. $Y^{t,\hat{\mu}',\alpha,i}$) and δY^i for $Y'^i - Y^i$. We, then, have

$$\delta Y^i_s \quad = \quad (y' - y) + \int_t^s \chi^1_u \delta Y^i_u \mathrm{d}u + \int_t^s \chi^2_u \delta Y^i_u \mathrm{d}B^i_u ,$$

for $s \geq t$, where χ^1 and χ^2 are given by

$$\chi^1_u := \frac{\bar{\lambda}_Y\left(u, X^i_u, Y^i_u\right) - \bar{\lambda}_Y\left(u, X^i_u, Y'^i_u\right)}{\delta Y_u}, \qquad \chi^2_u := \frac{\bar{\sigma}_Y\left(u, X^i_u, Y^i_u\right) - \bar{\sigma}_Y\left(u, X^i_u, Y'^i_u\right)}{\delta Y_u},$$

for $u \geq 0$, with $\bar{\lambda}_Y$ and $\bar{\sigma}_Y$ defined in Remark 1.2.1. From the Lipschitz property of λ_Y and σ_Y in Assumption A1, χ^1 and χ^2 are bounded. This is a linear SDE, whose solution is given by

$$\bar{Y}^i_s = (y' - y)\exp\left(\int_t^s \chi^2_u \mathrm{d}B^i_u - \int_t^s \left(\frac{1}{2}|\chi^2_u|^2 - \chi^1_u\right)\mathrm{d}u\right) \geq 0, \quad \mathbb{P}-\text{a.s.},$$

for $s \geq t$. Therefore, since $y \in \mathcal{Y}(t,\mu)$, we get

$$Y_T^{t,\mu,\alpha,y',i} \quad \geq \quad Y_T^{t,\mu,\alpha,y,i} \;\geq\; g_i\left(X_T^{t,\mu,\alpha,i}\right), \quad \mathbb{P}-\text{a.s.},$$

This is true for all $i \in \mathcal{V}_T^{t,\mu}$, therefore, $y' \in \mathcal{R}(t,\mu)$. □

From Proposition 1.2.5, the closure $\overline{\mathcal{R}(t,\mu)}$ of the reachability set is a half-line interval characterized by its lower bound. We, then, define the value function v as the infimum of $\mathcal{R}$:

$$\begin{aligned} v(t,\mu) &:= \inf \mathcal{R}(t,\mu) \\ &= \inf\Big\{y \in \mathbb{R} \;:\; \exists \alpha \in \mathcal{A}\,,\; Y_t^{t,\hat{\mu},\alpha,i} = y\,,\ \text{for } i \in V \\ &\qquad \text{and } Y_T^{t,\hat{\mu},\alpha,i} \geq g_i\left(X_T^{t,\mu,\alpha,i}\right),\ \text{for } i \in \mathcal{V}_T^{t,\mu},\ \mathbb{P}-\text{a.s.}\Big\}, \end{aligned} \tag{1.2.20}$$

for all $t \in [0,T]$ and $\mu = \sum_{i\in V}\delta_{(i,x_i)} \in E_d$, with the usual convention that $\inf(\emptyset) = +\infty$. We aim to provide an analytical characterization of the value function v.

Remark 1.2.2. *The value function v or the reachability set $\mathcal{R}$ might not be well defined in the case where the extinction of the alive population happens before T. In this case, we take the convention that the terminal condition is always satisfied if $\mathcal{V}_T^{t,\mu} = \emptyset$. In the sequel, we keep this convention for other constraints on $(X_\theta^{t,\mu,\alpha,i}, Y_\theta^{t,\hat{\mu},\alpha,i})$ with $\mathcal{V}_\theta^{t,\mu} = \emptyset$ and θ a stopping time.*

For $t \in [0,T]$, we define $\mathcal{A}_t$ as the subset of admissible controls independent of $\mathcal{F}_t^{\mathbb{P}}$. We, then, have the following result.

Proposition 1.2.6. *Under Assumption A1, the value function function v satisfies the following identity*

$$\begin{aligned} v(t,\mu) &= \inf\Big\{y \in \mathbb{R} \;:\; \exists \alpha \in \mathcal{A}_t\,,\; Y_t^{t,\hat{\mu},\alpha,i} = y\,,\ \textit{for } i \in V, \\ &\qquad \textit{and } Y_T^{t,\hat{\mu},\alpha,i} \geq g_i\left(X_T^{t,\mu,\alpha,i}\right),\ \textit{for } i \in \mathcal{V}_T^{t,\mu},\ \mathbb{P}-\textit{a.s.}\Big\}, \end{aligned} \tag{1.2.21}$$

for all $t \in [0,T]$ and $\mu = \sum_{i\in V}\delta_{(i,x_i)} \in E_d$.

Proof. Denote by $\tilde{v}(t,\mu)$ the right-hand side of (1.2.21). We obviously have $\tilde{v}(t,\mu) \geq v(t,\mu)$. We now prove the reverse inequality. Fix some $y \in \mathcal{R}(t,\mu)$ and $\alpha \in \mathcal{A}$ such that

$$Y_T^{t,\hat{\mu},\alpha,i} \quad \geq \quad g_i\left(X_T^{t,\mu,\alpha,i}\right), \qquad \text{for } i \in \mathcal{V}_T^{t,\mu}.$$

Then, we have

$$\mathbb{E}\Bigg[\sum_{j\in\mathcal{V}_T^{t,\mu}} \mathbb{1}_{Y_T^{t,\hat{\mu},\alpha,j}<g_j\left(X_T^{t,\mu,\alpha,j}\right)}\Big|\mathcal{F}_t^{\mathbb{P}}\Bigg] \quad = \quad 0\ .$$

Using the conditioning property from Theorem 1.2.2, we get

$$\mathbb{E}\Bigg[\sum_{j\in\mathcal{V}_T^{t,\mu}} \mathbb{1}_{Y_T^{t,\hat{\mu},\beta,j}<g_j\left(X_T^{t,\mu,\beta,j}\right)}\Bigg]\Bigg|_{\beta=\alpha^{t,\omega}} \quad = \quad 0 \text{ for } \mathbb{P}-a.a.\ \omega\in\Omega\ .$$

Therefore, we get $y \geq \tilde{v}(t,\mu)$ and $v(t,\mu) \geq \tilde{v}(t,\mu)$. □

We end this section with a new formulation of the function v.

Proposition 1.2.7. *Under Assumption A1, the value function function v satisfies the following identity*

$$\begin{aligned} v(t,\mu) \quad = \quad \inf\Big\{ & y\in\mathbb{R}\ :\ \exists\alpha\in\mathcal{A}_t\ ,\ \exists\hat{\mu}=\sum_{i\in V}\delta_{(i,x_i,y_i)}\in E_{d+1}\ \ \textit{such that} \\ & y_i \ \leq\ y\ ,\ \textit{for } i\in V,\ \textit{and } Y_T^{t,\hat{\mu},\alpha,i}\ \geq\ g_i(X_T^{t,\mu,\alpha,i})\ ,\ \textit{for } i\in\mathcal{V}_T^{t,\mu},\ \mathbb{P}-a.s.\Big\}\ , \end{aligned} \tag{1.2.22}$$

for all $t\in[0,T]$ and $\mu=\sum_{i\in V}\delta_{(i,x_i)}\in E_d$.

Proof. Denote by $\tilde{v}(t,\mu)$ the right hand side of (1.2.22). Since the set whose infimum is $v(t,\mu)$ is included in the one whose infimum is $\tilde{v}(t,\mu)$, we obviously have

$$\tilde{v}(t,\mu) \quad \leq \quad v(t,\mu)\ .$$

Fix now $y\in\mathbb{R}$ for which there exist $\alpha\in\mathcal{A}_t$ and $\hat{\mu}=\sum_{i\in V}\delta_{(i,x_i,y_i)}\in E_{d+1}$ such that

$$y_i \quad \leq \quad y\ , \qquad \text{for } i\in V\ ,$$

and

$$Y_T^{t,\hat{\mu},\alpha,i} \quad \geq \quad g_i(X_T^{t,\mu,\alpha,i})\ , \qquad \text{for } i\in\mathcal{V}_T^{t,\mu}\ .$$

Set $\bar{\mu}=\sum_{i\in V}\delta_{(i,x_i,y)}\in E_{d+1}$. By the comparison argument used in the proof of Proposition 1.2.5, we have

$$Y_T^{t,\bar{\mu},\alpha,i} \ \geq\ Y_T^{t,\hat{\mu},\alpha,i} \quad \geq \quad g_i(X_T^{t,\mu,\alpha,i})\ , \qquad \text{for } i\in\mathcal{V}_T^{t,\mu}\ .$$

Therefore, $y\geq v(t,\mu)$ and $\tilde{v}(t,\mu)\geq v(t,\mu)$. □

1.2.3 An example of application from fintech

Fintech is the contraction of the words finance and technology. It refers to recent technologies that allow for the improvement and automation of the delivery and use of financial services. The field emerged at the beginning of the 21st century and covered technologies used by established financial institutions. Since that time, the field has evolved to also include crypto-currencies, which are decentralised financial assets. These assets are based on blockchain technology. The main idea of this structure is to keep any new transaction registered in a chain by adding new blocks and sharing the extension of the original chain over the network so that every user keeps it in mind the transaction and can certify it. We refer to [122] for a description of how a blockchain-based crypto-currency works in the case of Bitcoin.

Due to the nature of this kind of asset, a fork can appear in the chain (see, *e.g*, [76]). In this case, the original asset is transformed into several ones. A natural question that arises is how to evaluate an option on crypto-currencies in this case. We present here the example of the super-replication of options on an asset that may fork and show that it is a particular case of the branching stochastic target presented above. We refer to the introduction of [88] for a list of financial markets where crypto-based derivatives are traded.

We consider a financial market where it is defined as a crypto-currency with price process $(S_t)_{t\in[0,T]}$. We suppose that the process S is a branching diffusion process and describes its dynamics. We first define the set $\mathcal{V}_t$ of alive particles at time $t \in [0,T]$ as previously done in Section 1.2.1. The initial condition for the process S is a constant ($S_0 > 0$). Assume the version $i \in \mathcal{I}$ of the crypto-currency is alive at time $t \in [0,T]$, dies at some random time $\tau_i \geq t$ and gives birth to k new versions $i0, \dots, i(k-1)$. The position at a time $s \geq \tau_i$ of the new crypto-currencies is given by

$$S^{i\ell}_{\tau_i} = S^{i}_{\tau_i-} \ , \tag{1.2.23}$$

$$\mathrm{d}S^{i\ell}_s = S^{i\ell}_s \left(b\mathrm{d}s + c\mathrm{d}B^{i\ell}_s\right) \ , \tag{1.2.24}$$

for $\ell = 0, \dots, k-1$ and $s \geq \tau_i$ such that version $i\ell$ is alive at time s. Here b and c are two positive constants.

In addition to that asset, we assume that there exists on the market a non-risky asset S^0 with deterministic interest rate $r > 0$ and with initial condition $S^0_0 = 1$, *i.e.*, $S^0_t = e^{rt}$ for $t \in [0,T]$.

An investment strategy consists in a process $\pi = (\pi^i_t)_{t\in[0,T], i\in\mathcal{I}}$ of $\mathbb{F}$-predictable processes valued in $[0,1]$, where π^i_t represents the proportion of the wealth invested in the version S^i of the crypto-currency. We denote by $\mathcal{A}$ the set of such strategies. For $\pi \in \mathcal{A}$, we also denote by $V^{V_0,\pi}$ the self financing wealth process related to the initial capital V_0 and strategy π. According to (1.2.23)-(1.2.24), this process is given by

$$V^{V_0,\pi,i\ell}_{\tau_i} = V^{V_0,\pi,i}_{\tau_i-} \ , \tag{1.2.25}$$

$$\mathrm{d}V^{V_0,\pi,i\ell}_s = V^{V_0,\pi,i\ell}_s \left(((b-r)\pi^{i\ell}_s + r)\mathrm{d}s + c\pi^{i\ell}_s \mathrm{d}B^{i\ell}_s\right) \ , \tag{1.2.26}$$

for $\ell = 0, \dots, k-1$ and $s \geq \tau_i$ such that version $i\ell$ is alive at time s.

We, then, consider a financial derivative on the asset S that consists of a Put Option but with a strike K_i depending on the version of the crypto-currency S. Such a product can express the need to hedge against a decrease in the value of the asset S that depends on the branch.

The computation of the super-replication problem leads to solving the following stochastic target problem

$$w_0 = \inf\Big\{\nu \in \mathbb{R}_+ \ : \ \exists \pi \in \mathcal{A} \ , \ V^{\nu,\pi,i}_T \ \geq \ (K_i - S^i_T)_+ + \kappa \ , \quad \text{for } i \in \mathcal{V}_T, \ \mathbb{P}\text{a.s.}\Big\} \ ,$$

where κ is a positive constant representing some friction of the market. We next modify this problem to satisfy our assumptions. For that, we first define the processes

$$\begin{aligned} Y_t^{y,\pi,i} &:= \log\left(V_t^{e^y,\pi,i}\right) \\ X_t^i &:= \log\left(S_t^i\right) , \end{aligned}$$

for $t \in [0,T]$ and $i \in \mathcal{V}_t$. From (1.2.23)-(1.2.24) and (1.2.25)-(1.2.26), we get

$$X_{\tau_i}^{i\ell} = X_{\tau_i-}^i , \qquad \mathrm{d}X_s^{i\ell} = \left(b - \frac{c^2}{2}\right)\mathrm{d}s + c\mathrm{d}B_s^{i\ell} , \tag{1.2.27}$$

$$Y_{\tau_i}^{y,\pi,i\ell} = Y_{\tau_i-}^{y,\pi,i} , \qquad \mathrm{d}Y_s^{y,\pi,i\ell} = \left((b-r)\pi_s^{i\ell} - \frac{1}{2}c^2\left(\pi_s^{i\ell}\right)^2 + r\right)\mathrm{d}s + c\pi_s^{i\ell}\mathrm{d}B_s^{i\ell} , \tag{1.2.28}$$

for $\ell = 0, \ldots, k-1$ and $s \geq \tau_i$ such that version $i\ell$ is alive at time s. We observe that the dynamics of the processes Y and X satisfy Assumption A1. We also define the functions g as

$$g_i(x) := \log\left((K_i - e^x)_+ + \kappa\right) , \quad \text{for } (x,i) \in \mathbb{R} \times \mathcal{I} ,$$

which satisfies Assumption A2. Finally, we define the optimal value

$$v_0 := \inf\left\{ y \in \mathbb{R} \ : \ \exists \pi \in \mathcal{A} ,\ Y_T^{y,\pi,i} \geq g_i(X_T^i) , \quad \text{for } i \in \mathcal{V}_T ,\ \mathbb{P}-\text{a.s.}\right\} ,$$

a special case of (1.2.20). We notice that the optimal value w_0 is related to v_0 by

$$w_0 = \exp(v_0) .$$

We suppose that $\bar{K} := \sup_{i\in\mathcal{I}} K_i < +\infty$. The value function v related to v_0 is, then, bounded. Indeed, by taking the initial condition $t \in [0,T]$ and $y = -r(T-t) + \log(\bar{K}+\kappa)$ and the control $\pi_t^i = 0$ for $i \in \mathcal{I}$ and $t \in [0,T]$, we get from (1.2.28) that

$$Y_T^{t,\hat{\mu},\pi,i} \geq g_i(X_T^{t,\mu,i}) , \quad \text{for } i \in \mathcal{V}_T^{t,\mu} ,$$

for $\mu = \sum_{i\in V} \delta_{(i,x^i)} \in E_d$ and $\mu = \sum_{i\in V} \delta_{(i,x^i,y)} \in E_{d+1}$. Therefore,

$$v(t,\mu) \leq -r(T-t) + \log(\bar{K}+\kappa) , \quad (t,\mu) \in [0,T] \times E_d .$$

Moreover, for any $y \in \mathcal{R}(t,\mu)$ and π the related admissible control, we have

$$\left((b-r)\pi_s^{i\ell} - \frac{1}{2}c^2(\pi_s^{i\ell})^2 + r\right) \leq \frac{1}{2}\left(\frac{b-r}{c}\right)^2 + r .$$

Therefore, we get

$$y + \left(\left(\frac{b-r}{c}\right)^2 + r\right)(T-t) \geq \mathbb{E}\left[Y_T^{t,\hat{\mu},\pi,i}\right] \geq \mathbb{E}\left[g_i(X_T^{t,\mu,\pi,i})\right] \geq \log(\kappa) .$$

This implies that

$$v(t,\mu) \geq -\left(\left(\frac{b-r}{c}\right)^2 + r\right)(T-t) + \log(\kappa) , \quad \text{for } (t,\mu) \in [0,T] \times E_d .$$

In particular, v satisfies the growth condition (1.4.72). If we suppose also that $r = 0$ and $g_i = 0$ for $i \in \mathcal{I}$ of the form $i = i_1 \cdots i_n$ with $i_\ell \geq I$ for some ℓ where I is a given bound, , then, v also satisfies condition (1.4.71). Since this model satisfies Assumption A4 below, the comparison Theorem 1.4.6 holds in this case.

1.3 Dynamic programming

1.3.1 Measurable selection

In establishing a dynamic programming principle, we need an admissible control that is built as the concatenation of admissible controls depending on the position of the branching processes at an intermediary time. For this end, we use a *measurable selection approach.*

Let $\mathcal{U}$ be the *target set* defined by

$$\mathcal{U}(t,\hat{\mu}) \quad := \quad \Big\{\alpha \in \mathcal{A}_t \ : \ Y_T^{t,\hat{\mu},\alpha,i} \ \geq \ g_i(X_T^{t,\mu,\alpha,i}) \ , \ \text{for } i \in \mathcal{V}_T^{t,\mu} \ , \ \mathbb{P}-\text{a.s.}\Big\} \ ,$$

for $(t,\hat{\mu}) \in [0,T] \times E_{d+1}$ with $\hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y_i)}$ and $\mu = \sum_{i\in V} \delta_{(i,x_i)} \in E_d$. Let $S := [0,T] \times E_{d+1}$ and

$$D \quad := \quad \big\{(t,\hat{\mu}) \in S \ : \ \mathcal{U}(t,\hat{\mu}) \neq \emptyset\big\} \ .$$

We aim to exhibit a function that associates to each $(t,\hat{\mu}) \in D$ a control $\alpha \in \mathcal{U}(t,\hat{\mu})$ in a measurable way.

We denote by $\mathcal{P}(S)$ the set of probability measures on $(S, \mathcal{B}([0,T]) \otimes \mathcal{B}(E_{d+1}))$ and we endow $\mathcal{A}$ with the Borel σ-algebra $\mathcal{B}(\mathcal{A})$ related to the distance $d_{A,\mathcal{I}}$

$$d_{A,\mathcal{I}} : (\alpha,\alpha') \quad \mapsto \quad \sum_{i\in\mathcal{I}} \frac{1}{2^{|i|}} \wedge \mathbb{E}\int_0^T d_A\left(\alpha_s^i, \alpha_s'^i\right) \mathrm{d}s \ .$$

We, then, have the following measurable selection result.

Lemma 1.3.1. *Suppose that Assumptions A1 and A2 hold. For each $\nu \in \mathcal{P}(S)$, there exists a measurable function $\phi_\nu : (D, \mathcal{B}(D)) \to (\mathcal{A}, \mathcal{B}(\mathcal{A}))$ such that*

$$\phi_\nu(t,\hat{\mu}) \in \mathcal{U}(t,\hat{\mu}) \ \textit{for } \nu\textit{-a.e. } (t,\hat{\mu}) \in D \ .$$

Proof. The set S, endowed with the product σ-algebra $\mathcal{B}([0,T]) \otimes \mathcal{B}(E_{d+1})$, is a Borel space as a product of Borel spaces. Moreover, $\mathcal{A}$, equipped with $\mathcal{B}(\mathcal{A})$, forms a Borel space as a countable product of Polish spaces. The assertion that the set of predictable processes is Polish concerning the L^1 distance can be proved by referencing, for instance, [4, Theorems 13.6 and 11.18]. Although these theorems are originally stated for real-valued processes, their extension to Polish space-valued processes is possible given that their applicability only relies on completeness.

Let C and $\bar{C}$ be as follows

$$\begin{aligned} C \quad &:= \quad \Big\{(t,\hat{\mu},\alpha) \in S \times \mathcal{A} \ : \ \alpha \in \tilde{\mathcal{U}}(t,\hat{\mu})\Big\} \ , \\ \bar{C} \quad &:= \quad \{(t,\hat{\mu},\alpha) \in S \times \mathcal{A} \ : \ \alpha \in \mathcal{A}_t\} \ . \end{aligned}$$

with $\tilde{\mathcal{U}}(t,\hat{\mu}) := \Big\{\alpha \in \mathcal{A} \ : \ Y_T^{t,\hat{\mu},\alpha,i} \ \geq \ g_i(X_T^{t,\mu,\alpha,i}) \ , \ \text{for } i \in \mathcal{V}_T^{t,\mu} \ , \ \mathbb{P}-\text{a.s.}\Big\}$ for $(t,\hat{\mu}) \in S$.

From Proposition 1.2.4 and Assumption A2, C is closed and a fortiori a Borel subset of $S \times \mathcal{A}$. Let $\{\psi_\ell\}_\ell$ (resp. $\{\phi_k\}_k$) be a countable bounded family of functions from Ω (resp. $\mathcal{A}$) to $\mathbb{R}$ generating the Borel σ-algebra. We have that $\bar{C}$ belongs to the σ-algebra generated by the Borel sets

$$\Big\{(t,\hat{\mu},\alpha) \in S \times \mathcal{A} : \mathbb{E}\left[\psi_\ell\left((B^i_{\cdot\wedge t}, Q^i_{\cdot\wedge t})_{i\in\mathcal{I}}\right)\phi_k(\alpha)\right] = \mathbb{E}\left[\psi_\ell\left((B^i_{\cdot\wedge t}, Q^i_{\cdot\wedge t})_{i\in\mathcal{I}}\right)\right]\mathbb{E}\left[\phi_k(\alpha)\right]\Big\}\,.$$

Hence, $\bar{C}$ is Borel, thus $C \cap \bar{C}$ is a Borel set. The rest of the proof is a standard argument. For completeness, we briefly recall it.

Since $C \cap \bar{C}$ is a Borel set we get by [19, Propositions 7.36 and 7.49] that there exists an analytically measurable function $\phi : D \to \mathcal{A}$ such that

$$\{(t,\hat{\mu},\phi(t,\hat{\mu}))\ :\ (t,\hat{\mu}) \in S\} \quad \subseteq \quad C \cap \bar{C}\,.$$

Fix $\nu \in \mathcal{P}(S)$ and denote by $\mathcal{B}_\nu(S)$ the completion of the Borel σ-algebra $\mathcal{B}(S)$ under ν. From [19, Corollary 7.42.1], ϕ is universally measurable and, therefore, there exists a Borel measurable map ϕ_ν such that $\phi_\nu(t,\hat{\mu}) = \phi(t,\hat{\mu})$ for ν-almost every $(t,\hat{\mu}) \in S$.

□

1.3.2 Dynamic programming principle

We are now able to state the DPP as follows.

Theorem 1.3.3. *Under Assumptions A1 and A2, the value function satisfies*

$$\begin{aligned} v(t,\mu) \quad = \quad \inf\Big\{ & y \in \mathbb{R}\ :\ \exists \alpha \in \mathcal{A}\,,\ \exists \hat{\mu} = \sum_{i\in V}\delta_{(i,x_i,y_i)} \in E_{d+1} \ \ \textit{such that} \\ & y_i \ \leq \ y\ ,\ \textit{for } i \in V \ \textit{and } Y^{t,\hat{\mu},\alpha,i}_\theta \ \geq \ v\left(\theta, \delta_{\left(i, X^{t,\mu,\alpha,i}_\theta\right)}\right)\ , \quad \textit{for } i \in \mathcal{V}^{t,\mu}_\theta\ ,\ \mathbb{P}-\textit{a.s.}\Big\}\,, \end{aligned} \tag{1.3.29}$$

for any $(t,\mu) \in [0,T] \times E_d$ and $\theta \in \mathcal{T}_{[t,T]}$.

Proof. We first define the *reachability sets* by

$$\mathcal{Y}(t,\mu) \quad := \quad \Big\{(y_i)_{i\in V} \in \mathbb{R}^V \quad :\ \mathcal{U}(t,\hat{\mu}) \neq \emptyset \text{ with } \hat{\mu} = \sum_{i\in V}\delta_{(i,x_i,y_i)}\Big\}\,,$$

and

$$\begin{aligned} \mathcal{Y}^\theta(t,\mu) \quad := \quad \Big\{ & (y_i)_{i\in V} \in \mathbb{R}^V\ :\ \exists \alpha \in \mathcal{A} \ \text{ such that} \\ & Y^{t,\hat{\mu},\alpha,i}_\theta \geq v\left(\theta, \delta_{\left(i, X^{t,\mu,\alpha,i}_\theta\right)}\right)\ ,\ \text{for } i \in \mathcal{V}^{t,\mu}_\theta\ ,\ \mathbb{P}-\text{a.s. with } \hat{\mu} = \sum_{i\in V}\delta_{(i,x_i,y_i)} \in E_{d+1}\Big\}\,, \end{aligned}$$

for $t \in [0,T]$, $\mu = \sum_{i\in V}\delta_{(i,x_i)} \in E_d$ and $\theta \in \mathcal{T}_{[t,T]}$. Fix now $t \in [0,T]$ and $\mu = \sum_{i\in V}\delta_{(i,x_i)} \in E_d$ and denote by $v_\theta(t,\mu)$ the right hand side of (1.3.29).

To prove $v(t,\mu) \geq v_\theta(t,\mu)$, we show that $\mathcal{Y}(t,\mu) \subseteq \mathcal{Y}^\theta(t,\mu)$. Fix $(y_i)_{i\in V} \in \mathcal{Y}(t,\mu)$. We have, by definition of $/Yc(t,\mu)$, that there exists $\alpha \in \mathcal{A}$ such that

$$Y_T^{t,\hat{\mu},\alpha,j} \geq g_j\left(X_T^{t,\mu,\alpha,j}\right), \qquad \text{for } j \in \mathcal{V}_T^{t,\mu},\ \mathbb{P}-\text{a.s.}\ .$$

Taking the conditional expectation given $\mathcal{F}_\theta^{\mathbb{P}}$, we obtain that

$$\mathbb{E}\left[\sum_{j\in\mathcal{V}_T^{t,\mu}} \mathbb{1}_{Y_T^{t,\hat{\mu},\alpha,j} < g_j\left(X_T^{t,\mu,\alpha,j}\right)} \Big| \mathcal{F}_\theta^{\mathbb{P}}\right] = 0, \qquad \mathbb{P}-\text{a.s.}\ .$$

Combining this with the conditioning property from Theorem 1.2.2, we have

$$F\left(\theta(\omega), \hat{Z}_{\theta(\omega)}^{t,\hat{\mu},\alpha}(\omega), \alpha^{\theta(\omega),\omega}\right) = 0\ , \qquad \mathbb{P}(d\omega) - \text{a.s.}\ , \tag{1.3.30}$$

where

$$F(s,\hat{z},\beta) \quad := \quad \mathbb{E}\left[\sum_{j\in\mathcal{V}_T^{s,z}} \mathbb{1}_{Y_T^{s,\hat{z},\beta,j} < g_j\left(X_T^{s,z,\beta,j}\right)}\right]\ ,$$

for all $s \in [0,T]$, $\hat{z} \in E_{d+1}$ with $\hat{z} = \sum_{i\in\tilde{V}} \delta_{(i,\tilde{x}_i,\tilde{y}_i)}$, $z = \sum_{i\in\tilde{V}} \delta_{(i,\tilde{x}_i)}$, and $\beta \in \mathcal{A}$. Therefore, using the identity

$$\hat{Z}_T^{s,\hat{z}_1+\hat{z}_2,\beta} \quad = \quad \hat{Z}_T^{s,\hat{z}_1,\beta} + \hat{Z}_T^{s,\hat{z}_2,\beta}\ ,$$

for $\hat{z}_1, \hat{z}_2 \in E_{d+1}$ such that $\hat{z}_1 + \hat{z}_2 \in E_{d+1}$, we observe that the function F is additive w.r.t. its second argument, *i.e.*,

$$F(s,\hat{z}_1+\hat{z}_2,\beta) \quad = \quad F(s,\hat{z}_1,\beta) + F(s,\hat{z}_2,\beta)\ .$$

Together with (1.3.30), the previous remarks entail that

$$F\left(\theta(\omega), \delta_{(i,\hat{X}_{\theta(\omega)}^{t,\hat{\mu},\alpha,i}(\omega))}, \alpha^{\theta(\omega),\omega}\right) = 0\ , \qquad \text{for } i \in \mathcal{V}_{\theta(\omega)}^{t,\mu}(\omega)\ , \mathbb{P}(d\omega) - \text{a.s.}$$

Given the definition of the value function v, we obtain that $Y_\theta^{t,\hat{\mu},\alpha,i} \geq v\left(\theta, \delta_{(i,X_\theta^{t,\mu,\alpha,i})}\right)$ for all $i \in \mathcal{V}_\theta^{t,\mu}$, $\mathbb{P}$-a.s.,

We now turn to the reverse inequality $v_\theta(t,\mu) \geq v(t,\mu)$ and prove that $\mathcal{Y}_\varepsilon^\theta(t,\mu) \subseteq \mathcal{Y}(t,\mu)$ for any $\varepsilon > 0$, where

$$\mathcal{Y}_\varepsilon^\theta(t,\mu) \quad := \quad \left\{(y_i+\varepsilon)_{i\in V}\ :\ (y_i)_{i\in V} \in \mathcal{Y}^\theta(t,\mu)\right\}\ .$$

Let $(y_i)_{i\in V} \in \mathcal{Y}^\theta(t,\mu)$ and $\alpha \in \mathcal{A}$ be such that $Y_\theta^{t,\hat{\mu},\alpha,i} \geq v\left(\theta, \delta_{\left(i,X_\theta^{t,\mu,\alpha,i}\right)}\right)$, for all $i \in \mathcal{V}_\theta^{t,\mu}$, $\mathbb{P}$-a.s. where $\hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y_i)}$. Fix now $\varepsilon > 0$ and set $\hat{\mu} = \sum_{i\in V} \delta_{(i,x_i,y_i)}$ and $\hat{\mu}_\varepsilon = \sum_{i\in V} \delta_{(i,x_i,y_i+\varepsilon)}$. From the definition of the value function and the strict monotonicity of the flow w.r.t. the initial value, we get $Y_\theta^{t,\hat{\mu},\alpha,i}(\omega) < Y_\theta^{t,\hat{\mu}_\varepsilon,\alpha,i}(\omega) \in \mathcal{Y}\left(\theta, \delta_{(i,X_\theta^{t,\mu,\alpha,i})}\right)(\omega)$ for all $i \in \mathcal{V}_\theta^{t,\mu}(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We consider now the probability measure ν induced on S by

$$\omega \mapsto \left(\theta, \hat{Z}_\theta^{t,\hat{\mu}_\varepsilon,\alpha}\right)(\omega) ,$$

and ϕ_ν the measurable map defined in Lemma 1.3.1. We have

$$Y_T^{u,\hat{\eta},\phi_\nu(u,\hat{\eta}),i} \;\; \geq \;\; g_i\left(X_T^{u,\eta,\phi_\nu(u,\hat{\eta}),i}\right) , \qquad \text{for } i \in \mathcal{V}_T,\ \mathbb{P}-\text{a.s.} \tag{1.3.31}$$

for $(u, \hat{\eta}) \in D \setminus N^\nu$, where N^ν is a ν-negligible subset of S and $\eta(\mathrm{d}i, \mathrm{d}x) := \int_{\mathbb{R}} \hat{\eta}(\mathrm{d}i, \mathrm{d}x, \mathrm{d}y)$. For $(u, \hat{\eta}) \in D \setminus N^\nu$, we denote by $N_{2,(u,\hat{\eta})}$ the negligible set of $\mathcal{F}^{\mathbb{P}}$ such that (1.3.31) holds on $\Omega \setminus N_{2,(u,\hat{\eta})}$. We now define

$$\hat{\Xi} \;:=\; \left(\theta, \hat{Z}_\theta^{t,\hat{\mu}_\varepsilon,\alpha}\right) \quad \text{ and } \quad \Xi \;:=\; \left(\theta, Z_\theta^{t,\mu,\alpha}\right)$$

together with the sets

$$N_1 \;:=\; \hat{\Xi}^{-1}\big(N^\nu\big) \quad \text{ and } \quad N_{2,\omega_1} \;:=\; N_{2,\hat{\Xi}}(\omega_1) , \quad \omega_1 \in N_1 .$$

We notice N_1 and N_{2,ω_1} are negligible sets of $\mathcal{F}^{\mathbb{P}}$. Therefore, from From (1.3.31), we get

$$Y_T^{\hat{\Xi}(\omega_1),\phi_\nu\left(\hat{\Xi}(\omega_1)\right),i}(\omega_2) \;\; \geq \;\; g_i\left(X_T^{\Xi(\omega_1),\phi_\nu\left(\hat{\Xi}(\omega_1)\right),i}(\omega_2)\right) , \qquad \text{for } i \in \mathcal{V}_T^{\Xi(\omega_1)}(\omega_2) ,$$

for all $\omega_1 \in N_1^c$ and $\omega_2 \in N_{2,\omega_1}^c$. Since $\phi_\nu(\hat{\Xi})$ is independent of $\mathcal{F}_\theta^{\mathbb{P}}$, the conditioning property from Theorem 1.2.2 implies that

$$\mathbb{E}\left[\sum_{j\in\mathcal{V}_T^{t,\mu}} \mathbb{1}_{Y_T^{t,\hat{\mu}_\varepsilon,\bar{\alpha},j} < g_j\left(X_T^{t,\mu,\bar{\alpha},j}\right)} \middle| \mathcal{F}_\theta^{\mathbb{P}}\right] \;\; = \;\; 0 ,$$

with $\bar{\alpha}$ given by

$$\bar{\alpha}^i(\omega) := \alpha^i(\omega)\mathbb{1}_{[0,\theta(\omega))} + \phi_\nu^i(\hat{\Xi}(\omega))(\omega)\mathbb{1}_{[\theta(\omega),T]} , \quad \text{if } \omega \in \Omega .$$

Therefore,

$$\mathbb{E}\left[\sum_{j\in\mathcal{V}_T^{t,\mu}} \mathbb{1}_{Y_T^{t,\hat{\mu}_\varepsilon,\bar{\alpha},j} < g_j\left(X_T^{t,\mu,\bar{\alpha},j}\right)}\right] \;\; = \;\; 0$$

and $(y_i + \varepsilon)_{i\in V} \in \mathcal{Y}(t, \mu)$.

□

1.4 PDE characterisation

1.4.1 Branching property

Conditionally to their birth, the living particles and their branches are independent in the uncontrolled case. In our case, this *branching property* is passed down to the value function in the following way.

Proposition 1.4.8 (Branching property). *Let Assumption A1 hold. The value function v satisfies*

$$v(t,\mu) \quad = \quad \max_{i\in V} v(t,\delta_{(i,x^i)})\ , \tag{1.4.32}$$

for any $(t,\mu=\sum_{i\in V}\delta_{(i,x^i)})\in[0,T]\times E_d$.

Proof. Proving $v(t,\mu)\geq \max_{i\in V} v(t,\delta_{(i,x^i)})$ comes to verify that $\mathcal{R}(t,\mu)\subseteq\bigcap_{j\in V}\mathcal{R}\left(t,\delta_{(j,x^j)}\right)$, *i.e.*, $\mathcal{R}(t,\mu)\subseteq\mathcal{R}\left(t,\delta_{(j,x^j)}\right)$ for each $j\in V$, with $\mathcal{R}$ as in (1.2.19). If $y\in\mathcal{R}(t,\mu)$, there exists α satisfying the constraints in T, $\mathbb{P}$-a.s. With this same α, zooming in on the sub-population generated by each $j\in V$, we must satisfy the condition of $\mathcal{R}\left(t,\delta_{(j,x^j)}\right)$. Therefore, $y\in\mathcal{R}\left(t,\delta_{(j,x^j)}\right)$.

Let j be the index that realizes the maximum in the righthand side of (1.4.32). The monotonicity property given by Proposition 1.2.5 implies $\mathcal{R}\left(t,\delta_{(j,x^j)}\right)\subseteq\mathcal{R}\left(t,\delta_{(i,x^i)}\right)$ for all $i\in V$. Then, if $y\in\mathcal{R}\left(t,\delta_{(j,x^j)}\right)$, let α^i be a control for $i\in V$ that meets the demand of $\mathcal{R}\left(t,\delta_{(i,x^i)}\right)$. To prove $y\in\mathcal{R}(t,\mu)$ we must exhibit a control that satisfies the requirements of such a set. Having a control α taken as α^i on the branches generated by each $i\in V$, we meet the conditions of $\mathcal{R}(t,\mu)$. Therefore, $\max_{i\in V} v(t,\delta_{(i,x^i)})=v(t,\delta_{(j,x^j)})\geq v(t,\mu)$ □

This result shows that it is enough to focus on the function $\bar v$ defined as follows

$$\bar v_i(t,x) \quad := \quad v(t,\delta_{(i,x)})\ ,$$

for $(i,t,x)\in\mathcal{I}\times[0,T]\times\mathbb{R}^d$. We provide in the next sections a PDE characterization of the function $\bar v$.

1.4.2 Dynamic programming equation

The equation on the parabolic interior

In a stochastic target problem, as we ask to hit a given target with probability one, we must degenerate along certain directions. Moreover, in this case, we aim at controlling also the uncertainty related to the possible branching. This property enables the characterization of the value function $\bar v$ as a solution to the following PDE

$$\min\left\{-\partial_t\bar v_i(t,x)+F\left(x,\bar v_i(t,x),D\bar v_i(t,x),D^2\bar v_i(t,x)\right)\ ;\ \bar v_i(t,x)-\sup_{0\leq k<\bar K}\bar v_{ik}(t,x)\right\} \quad = \quad 0 \tag{1.4.33}$$

for $(t,x)\in[0,T)\times\mathbb{R}^d$, where

$$\begin{aligned}\bar K &:= \sup\{k+1\in\mathbb{N}\ :\ p_k>0\}\ ,\\ F(\Theta) &:= \sup\left\{\lambda_Y(x,y,a)-\lambda(x,a)^\top p-\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,a)M\right)\ :\ a\in\mathcal{N}(x,y,p)\right\}\ ,\end{aligned}$$

with $\bar K=\infty$ in the case that $\{k\in\mathbb{N}\ :\ p_k>0\}$ is unbounded, for $\Theta:=(x,y,p,M)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$, and

$$\mathcal{N}(x,y,p):=\{a\in A\ :\ N^a(x,y,p)=0\}\quad\text{and}\quad N^a(x,y,p):=\sigma_Y(x,y,a)-\sigma(x,a)^\top p\ ,$$

for $x,p\in\mathbb{R}^d$ and $y\in\mathbb{R}$.

Since the control set A is not necessarily compact, the operator associated with this PDE may not be continuous. We, therefore, need to define a weak formulation of (1.4.33). For that, we introduce the relaxed semi-limits of F given by

$$F^*(\Theta) = \limsup_{\varepsilon\to 0,\Theta'\to\Theta} F_\varepsilon(\Theta') \quad \text{and} \quad F_*(\Theta) = \liminf_{\varepsilon\to 0,\Theta'\to\Theta} F_\varepsilon(\Theta')\ ,$$

where

$$F_\varepsilon(\Theta) := \sup\left\{\lambda_Y(x,y,a) - \lambda(x,a)^\top p - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,a)M\right)\ :\ a\in\mathcal{N}_\varepsilon(x,y,p)\right\}\ ,$$

for $\Theta = (x,y,p,M)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$, $\varepsilon\geq 0$, and

$$\mathcal{N}_\varepsilon(x,y,p) = \{a\in A\ :\ |N^a(x,y,p)|\leq\varepsilon\}\ ,$$

for $x,p\in\mathbb{R}^d$ and $y\in\mathbb{R}$. Observe that $(\mathcal{N}_\varepsilon)_{\varepsilon\geq 0}$ is non-decreasing so that

$$F_*(\Theta) = \liminf_{\Theta'\to\Theta} F_0(\Theta')\ . \tag{1.4.34}$$

Since some $\mathcal{N}_\varepsilon(x,y,p)$ may be empty, we shall use the standard convention $\sup\emptyset = -\infty$ all over this paper. For ease of notation, we also write $F\varphi(t,x)$ in place of $F(x,\varphi(t,x),D\varphi(t,x),D^2\varphi(t,x))$ for a regular function φ. We similarly use the notations $F^*\varphi$ and $F_*\varphi$.

As the value function may not be regular, we use the framework of discontinuous viscosity solutions. To this end, we define the lower- and upper-semicontinuous envelopes f_* and f^* of a locally bounded function $f:[0,T]\times\mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$ by

$$f_i^*(t,x) := \limsup_{\substack{(t',x')\to(t,x)\\ t'<T}} f_i(t',x') \quad \text{and} \quad f_{i,*}(t,x) := \liminf_{\substack{(t',x')\to(t,x)\\ t'<T}} f_i(t',x')\ , \tag{1.4.35}$$

for $(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}$. We are now able to define a viscosity solution to (1.4.33).

Definition 1.4.2. *Let $u:\ [0,T]\times\mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$ be a locally bounded function.*
(i) u is a viscosity supersolution *to (1.4.33) if for any $(t_0,x_0,i_0)\in[0,T)\times\mathbb{R}^d\times\mathcal{I}$, $\varphi_i\in C^{1,2}([0,T]\times\mathbb{R}^d)$, for $i\in\mathcal{I}$, and $\bar\varphi\in C^0([0,T]\times\mathbb{R}^d)$ such that*

$$\begin{aligned}\sup_{i\in\mathcal{I}}|\varphi_i(t,x)| &\leq \bar\varphi(t,x)\ , \qquad \text{for } (t,x)\in[0,T]\times\mathbb{R}^d\ ,\\ 0 = (u_{i_0,*}-\varphi_{i_0})(t_0,x_0) &= \min_{\mathcal{I}\times[0,T]\times\mathbb{R}^d}(u_{\cdot,*}-\varphi_\cdot)\ ,\end{aligned}$$

we have

$$\min\left\{-\partial_t\varphi_{i_0}(t_0,x_0)+F^*\varphi_{i_0}(t_0,x_0)\ ;\ \left(\varphi_{i_0}-\sup_{0\leq k<\bar K}\varphi_{i_0k}\right)(t_0,x_0)\right\}\ \geq\ 0\ .$$

(ii) u is a viscosity subsolution *to (1.4.33) if for any $(t_0,x_0,i_0)\in[0,T)\times\mathbb{R}^d\times\mathcal{I}$, $\varphi_i\in C^{1,2}([0,T]\times\mathbb{R}^d)$, for $i\in\mathcal{I}$, and $\bar\varphi\in C^0([0,T]\times\mathbb{R}^d)$ such that*

$$\begin{aligned}\sup_{i\in\mathcal{I}}|\varphi_i(t,x)| &\leq \bar\varphi(t,x)\ , \qquad \text{for } (t,x)\in[0,T]\times\mathbb{R}^d\ ,\\ 0 = (u^*_{i_0}-\varphi_{i_0})(t_0,x_0) &= \max_{\mathcal{I}\times[0,T]\times\mathbb{R}^d}(u^*_\cdot-\varphi_\cdot)\ ,\end{aligned}$$

we have

$$\min\left\{-\partial_t\varphi_{i_0}(t_0,x_0)+F_*\varphi_{i_0}(t_0,x_0)\ ;\ \left(\varphi_{i_0}-\sup_{0\leq k<\bar K}\varphi_{i_0k}\right)(t_0,x_0)\right\}\quad\leq\quad 0\ .$$

(iii) u is a viscosity solution *to* (1.4.33) *if it is both a viscosity sub and supersolution to* (1.4.33).

We notice that the definition of viscosity solution is slightly different from the classical one as we impose a bound in the label i for test functions.

Following [26], we introduce the continuity assumption on the kernel used to prove the subsolution property.

Assumption A3. *Let B be a subset of $\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$ such that $\mathcal{N}_0\neq\emptyset$ on B. Then, for every $\varepsilon>0$, $(x_0,y_0,p_0)\in int(B)$, and $a_0\in\mathcal{N}_0(x_0,y_0,p_0)$, there exists an open neighborhood B' of (x_0,y_0,p_0) and a locally Lipschitz map $\hat a$ defined on B' such that $|\hat a(x_0,y_0,p_0)-a_0|\leq\varepsilon$ and*

$$\hat a(x,y,p)\in\mathcal{N}_0(x,y,p)\ \ \textit{for all}\ (x,y,p)\in B'\ .$$

We are now able to state the following result.

Theorem 1.4.4. *Suppose that $\bar v$ is locally bounded on $[0,T]\times\mathbb{R}^d\times\mathcal{I}$.*

(i) Under Assumption A1, the value function $\bar v$ is a viscosity supersolution to (1.4.33)

(ii) If in addition Assumption A3 holds, $\bar v$ is a viscosity subsolution to (1.4.33)

Terminal condition

To get a complete characterization of the function $\bar v$, we need to add a terminal equation to (1.4.33). By the definition of the stochastic target problem, we have

$$\bar v_i(T,x)=g_i(x)\ , \tag{1.4.36}$$

for every $(x,i)\in\mathbb{R}^d\times\mathcal{I}$. The possible discontinuities of $\bar v$ might imply that $\bar v_*$ and $\bar v^*$ do not agree with the boundary condition (1.4.36). To get the proper terminal condition, we introduce the set-valued map

$$\mathbf{N}(x,y,p)=\{r\in\mathbb{R}^m\ :\ r=N^a(x,y,p)\ \text{ for some } a\in A\}\ ,$$

together with the signed distance function from its complement set $\mathbf{N}^c$ to the origin

$$\delta=\text{dist}(0,\mathbf{N}^c)-\text{dist}(0,\mathbf{N})\ ,$$

where dist stands for the Euclidean distance. Then, we have that

$$0\in\text{int}\mathbf{N}(x,y,p)\quad\Leftrightarrow\quad\delta(x,y,p)>0\ . \tag{1.4.37}$$

For simplicity of notations, we will write $\delta\varphi(x)$ for $\delta(x,\varphi(x),D\varphi(x))$ for a regular function φ. Under this notations, the terminal condition takes the following form

$$\min\left\{\ \bar v_i(T,x)-g_i(x)\ ;\ \delta\bar v_i(T,x)\ ;\ \left(\bar v_i-\sup_{0\leq k<\bar K}\bar v_{ik}\right)(T,x)\ \right\}\quad=\quad 0\ , \tag{1.4.38}$$

for $(x,i) \in \mathbb{R}^d \times \mathcal{I}$.

We define now a viscosity solution to (1.4.38).

Definition 1.4.3. *Let $u : [0,T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ be a locally bounded function.*
(i) u is a viscosity supersolution *to* (1.4.38) *if for any $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{I}$, $\varphi_i \in C^2(\mathbb{R}^d)$, for $i \in \mathcal{I}$, and $\bar\varphi \in C^0(\mathbb{R}^d)$ such that*

$$\begin{aligned} \sup_{i \in \mathcal{I}} |\varphi_i(x)| &\leq \bar\varphi(x) , \qquad \textit{for } x \in \mathbb{R}^d , \\ 0 = u_{i_0,*}(T,x_0) - \varphi_{i_0}(x_0) &= \min_{\mathcal{I}\times\mathbb{R}^d} (u_{\cdot,*}(T,\cdot) - \varphi_\cdot) , \end{aligned}$$

we have

$$\min\left\{ (\varphi_{i_0}(x_0) - g_i(x_0))\, \mathbb{1}_{F^*\varphi_{i_0}(x_0)<\infty} \ ; \ \delta^*\varphi_{i_0}(x_0) \ ; \ \varphi_{i_0}(T,x_0) - \sup_{0\leq k<\bar K} \varphi_{i_0 k}(T,x_0) \right\} \geq 0 .$$

(ii) u is a viscosity subsolution *solution to* (1.4.38) *if for any $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{I}$, $\varphi_i \in C^2(\mathbb{R}^d)$, for $i \in \mathcal{I}$, and $\bar\varphi \in C^0(\mathbb{R}^d)$ such that*

$$\begin{aligned} \sup_{i \in \mathcal{I}} |\varphi_i(x)| &\leq \bar\varphi(x) , \qquad \textit{for } x \in \mathbb{R}^d , \\ 0 = u^*_{i_0}(T,x_0) - \varphi_{i_0}(x_0) &= \max_{\mathcal{I}\times\mathbb{R}^d} (u^*_\cdot(T,\cdot) - \varphi_\cdot) , \end{aligned}$$

we have

$$\min\left\{ \varphi_{i_0}(x_0) - g_{i_0}(x_0); \ \delta_*\varphi_{i_0}(x_0) \ ; \ \varphi_{i_0}(T,x_0) - \sup_{0\leq k<\bar K} \varphi_{i_0 k}(T,x_0) \right\} \leq 0 .$$

(iii) u is a viscosity solution *to* (1.4.38) *if it is both a viscosity sub and supersolution to* (1.4.38).

The terminal viscosity property is stated as follows.

Theorem 1.4.5. *Suppose that $\bar v$ is locally bounded on $[0,T] \times \mathbb{R}^d \times \mathcal{I}$.*

(i) Under Assumptions A1 and A2, $\bar v$ is a viscosity supersolution to (1.4.38).

(ii) If in addition Assumption A3 holds, $\bar v$ is a viscosity subsolution to (1.4.38).

1.4.3 Viscosity properties on $[0,T) \times \mathbb{R}^d \times \mathcal{I}$

Viscosity supersolution property

Fix $(i_0, t_0, x_0) \in \mathcal{I} \times [0,T) \times \mathbb{R}^d$, $\varphi \in C^0([0,T] \times \mathbb{R}^d)$ and $\varphi_i \in C^{1,2}([0,T] \times \mathbb{R}^d)$, for $i \in \mathcal{I}$, such that

$$\sup_i |\varphi_i| \leq \varphi \tag{1.4.39}$$

and

$$0 = (\bar v_{i_0,*} - \varphi_{i_0})(t_0,x_0) = \min_{(i,t,x)\in\mathcal{I}\times[0,T]\times\mathbb{R}^d} (\bar v_{i,*} - \varphi_i)(t,x) . \tag{1.4.40}$$

Without loss of generality, we can assume this minimum to be strict in (t,x) once fixed i_0.

Step 1. We first prove that $\varphi_{i_0}(t_0,x_0) - \sup_{0\le\ell\le k-1}\varphi_{i_0k}(t_0,x_0) \ge 0$, for $k\in\mathbb{N}$ such that $p_k>0$. Let (t_n,x_n) be a sequence in $[0,T]\times\mathbb{R}^d$ such that

$$(t_n,x_n)\to(t_0,x_0) \text{ and } \bar{v}_{i_0}(t_n,x_n)\to\bar{v}_{i_0,*}(t_0,x_0) \text{ as } n\to\infty\,.$$

Set $y_0:=\varphi_{i_0}(t_0,x_0)$, $\hat{x}_0:=(x_0,y_0)$, $y_n:=\bar{v}_{i_0}(t_n,x_n)+1/n$ and $\hat{x}_n:=(x_n,y_n)$. Define the stopping time

$$\theta_n:=\inf\{s\ge t_n\ :\ Q^{i_0}((t_n,s]\times\mathbb{N})\ge 1\}$$

and the random variable k_n such that $Q^{i_0}((t_n,\theta_n]\times\{k_n\})=1$. From Theorem 1.3.3, the continuity of the trajectories, and since $y_n>\bar{v}_{i_n}(t_n,x_n)$ there exists $\alpha^n\in\mathcal{A}$ such that

$$Y_{\theta_n}^{t_n,\delta_{(i_0,\hat{x}_n)},\alpha^n,i_0} \ \ge\ \max_{0\le\ell\le k_n-1}\bar{v}_{i_0\ell}\left(\theta_n,X_{\theta_n}^{t_n,\delta_{(i_0,x_n)},\alpha^n,i_0}\right)\ \ge\ \max_{0\le\ell\le k_n-1}\varphi_{i_0\ell}\left(\theta_n,X_{\theta_n}^{t_n,\delta_{(i_0,x_n)},\alpha^n,i_0}\right)\,,$$

on $\{\theta_n\le T\}$, where the previous processes are defined at θ_n according to the extension given by Remark 1.2.1. To alleviate the notation, we shall denote $X_t^{n,i}:=X_t^{t_n,\delta_{(i_0,x_n)},\alpha^n,i}$ and $Y_t^{n,i}:=Y_t^{t_n,\delta_{(i_0,x_n,y_n)},\alpha^n,i}$, for $n\ge1$ and $t\in[t_n,T]$. Therefore, we get

$$\begin{aligned}0 &= \mathbb{E}\left[\mathbb{1}_{Y_{\theta_n}^{n,i_0}<\max_{0\le\ell\le k_n-1}\varphi_{i_0\ell}\left(\theta_n,X_{\theta_n}^{n,i_0}\right)}\right]\\ &= \mathbb{E}\left[\int_{(t_n,T]\times\mathbb{N}}\mathbb{1}_{Y_s^{n,i_0}<\max_{0\le\ell\le k-1}\varphi_{i_0\ell}\left(s,X_s^{n,i_0}\right)}Q^{i_0}(\mathrm{d}s,\mathrm{d}k)\right]\,.\end{aligned}$$

As Q^{i_0} has intensity $\gamma\sum_{k\ge0}p_k\delta_k$, we obtain

$$0 \ =\ \mathbb{E}\left[\int_{t_n}^T\sum_{k\ge0}\mathbb{1}_{Y_s^{n,i_0}<\max_{0\le\ell\le k-1}\varphi_{i_0\ell}\left(s,X_s^{n,i_0}\right)}\gamma p_k\mathrm{d}s\right]\,,$$

which means

$$\int_{t_n}^T\mathbb{E}\left[\mathbb{1}_{Y_s^{n,i_0}<\max_{0\le\ell\le k-1}\varphi_{i_0\ell}\left(s,X_s^{n,i_0}\right)}\right]\mathrm{d}s \ =\ 0\,,$$

for all $k\ge1$ such that $p_k>0$. We, therefore, get

$$\mathbb{E}\left[\mathbb{1}_{Y_s^{n,i_0}<\max_{0\le\ell\le k-1}\varphi_{i_0\ell}\left(s,X_s^{n,i_0}\right)}\right] \ =\ 0\,, \tag{1.4.41}$$

for Lebesgue almost all $s\in[t_n,T]$. Since the process $Y^{n,i_0}-\max_{0\le\ell\le k-1}\varphi_{i_0\ell}\left(\cdot,X^{n,i_0}\right)$ is continuous, we achieve

$$y_n \ \ge\ \max_{0\le\ell\le k-1}\varphi_{i_0\ell}(t_n,x_n)\,,$$

for all $k\ge1$ such that $p_k>0$. Sending n to infinity gives the result.

Step 2. We now prove that

$$-\frac{\partial \varphi_{i_0}}{\partial t}(t_0, x_0) + F^* \varphi_{i_0}(t_0, x_0) \quad \geq \quad 0 \, .$$

Assume to the contrary that

$$(-\partial_t \varphi_{i_0} + F^* \varphi_{i_0})(t_0, x_0) = -2\eta \, ,$$

for some $\eta > 0$. By definition of F^* , we may find $\varepsilon \in (0, T - t_0)$, such that

$$\begin{aligned} -\partial_t \varphi_{i_0}(t,x) + \lambda_Y(x,y,a) - L^a \varphi_{i_0}(t,x) \leq -\eta \qquad & \text{for all } a \in \mathcal{N}_\varepsilon(x, \varphi_{i_0}(t,x), D\varphi_{i_0}(t,x)) \\ & \text{and } (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \text{ such that} \\ & (t,x) \in B_\varepsilon(t_0,x_0) \text{ and } |y - \varphi_{i_0}(t,x)| \leq \varepsilon \, , \end{aligned} \tag{1.4.42}$$

where $B_\varepsilon(t_0, x_0)$ denotes the ball of radius ε around (t_0, x_0). Let $\partial_p B_\varepsilon(t_0, x_0) = \{t_0 + \varepsilon\} \times \mathrm{cl}(B_\varepsilon(t_0, x_0)) \cup [t_0, t_0 + \varepsilon) \times \partial B_\varepsilon(x_0)$ denote the parabolic boundary of $B_\varepsilon(t_0, x_0)$ and observe that

$$\zeta = \min_{\partial_p B_\varepsilon(t_0,x_0)} (\bar{v}_{i_0,*} - \varphi_{i_0}) > 0 \, , \tag{1.4.43}$$

since (t_0, x_0) is a strict minimizer of $\bar{v}_{i_0,*} - \varphi_{i_0}$ on $[0, T) \times \mathbb{R}^d$.

Step 3. We now show that (1.4.42) and (1.4.43) lead to a contradiction to (1.3.29). Let (t_n, x_n) in $[0, T] \times \mathbb{R}^d$ such that

$$(t_n, x_n) \to (t_0, x_0) \ \text{ and } \ \bar{v}_{i_0}(t_n, \ x_n) \to \bar{v}_{i_0,*}(t_0, x_0) \ \text{ as } n \to \infty \, .$$

We set y_0, $\hat{x}_0$, y_n, and $\hat{x}_n$ as in Step 1 and notice that

$$\beta_n := y_n - \varphi_{i_0}(t_n, x_n) \quad \xrightarrow[n \to +\infty]{} \quad 0 \, . \tag{1.4.44}$$

From the definition of the value function and the fact that $y_n > \bar{v}_{i_0}(t_n, \ x_n)$ for each $n \geq 1$, there exists some α^n in $\mathcal{A}$ such that $Y_T^{t_n, \delta_{(i_0, x_n, y_n)}, \alpha^n, i} \geq g_i \left(X_T^{t_n, \delta_{(i_0, x_n)}, \alpha^n, i} \right)$ for all $i \in \mathcal{V}_T^{t_n, \delta_{(i_0,x_n)}, \alpha^n}$. To alleviate the notation, we shall use the notation $X_t^{n,i}$ and $Y_t^{n,i}$ as in Step 1, together with $\mathcal{V}_t^n := \mathcal{V}_t^{t_n, \delta_{(i_0,x_n)}, \alpha^n}$, for $n \geq 1$ and $t \in [t_n, T]$. Define the following stopping times

$$\begin{aligned} \tau_n \quad &:= \quad \inf\{s \geq t_n \ : \ \exists i \in \mathcal{V}_s^n \ , \ \left(s, X_s^{n,i}\right) \notin B_\varepsilon(t_0, x_0)\} \, , \\ \tau_n^\varepsilon \quad &:= \quad \inf\{s \geq t_n \ : \ \exists i \in \mathcal{V}_s^n \ , \ |Y_s^{n,i} - \varphi_i\left(s, X_s^{n,i}\right)| \geq \varepsilon\} \, , \\ \tau_n^r \quad &:= \quad \inf\{s \geq t_n \ : \ Q^{i_0}((t_n, s] \times \mathbb{N}) = 1 \ \} \, , \\ \theta_n \quad &:= \quad \tau_n \wedge \tau_n^\varepsilon \wedge \tau_n^r \, . \end{aligned}$$

Consider the following quantitites

$$\begin{aligned} A_n \quad &:= \quad \left\{ s \in [t_n, \theta_n) \ : \ -\partial_t \varphi_{i_0}(s, X_s^{n,i_0}) + \lambda_Y(X_s^{n,i_0}, Y_s^{n,i_0}, \alpha_{i_0}^n) - L^{\alpha_{i_0}^n} \varphi_{i_0}(s, X_s^{n,i_0}) > -\eta \right\} \quad (1.4.45) \\ \psi_s^n \quad &:= \quad N^{\alpha_{i_0}^n}(X_s^{n,i_0}, D\varphi_{i_0}(s, X_s^{n,i_0})) \, . \end{aligned}$$

We remark that (1.4.42) implies

$$|\psi_s^n| \quad > \quad \varepsilon \quad \text{for } s \in A_n \ . \tag{1.4.46}$$

Therefore, from Theorem 1.3.3, we obtain that

$$Y_{t\wedge\theta_n}^{n,i} \quad \geq \quad \bar{v}_i\left(t\wedge\theta_n, X_{t\wedge\theta_n}^{n,i}\right)\ , \qquad \text{for } i \in \mathcal{V}_{t\wedge\theta_n}^n,\ t\in[t_n,T]\ ,$$

and, since $\bar{v}_i \geq \bar{v}_{i,*} \geq \varphi_i$,

$$Y_{\theta_n\wedge t}^{n,i} \geq \varphi_i\left(\theta_n\wedge t, X_{\theta_n}^{n,i}\wedge t\right)\ , \qquad \text{for } i\in\mathcal{V}_{\theta_n}^n\ . \tag{1.4.47}$$

Using the definition of ζ in (1.4.43), the one of θ_n, and the continuity of the trajectories, we get

$$\begin{aligned} Y_{t\wedge\theta_n}^{n,i_0} \quad &\geq \quad \varphi_{i_0}\left(t\wedge\theta_n, X_{t\wedge\theta_n}^{n,i_0}\right) + (\zeta\mathbb{1}_{\{\theta_n=\tau_n\}} + \varepsilon\mathbb{1}_{\{\tau_n^\varepsilon=\theta_n\}\cap\{\theta_n<\tau_n\}})\mathbb{1}_{\{\theta_n\leq t\}\cap\{\theta_n<\tau_n^r\}} \\ &\geq \quad \varphi_{i_0}\left(t\wedge\theta_n, X_{t\wedge\theta_n}^{n,i_0}\right) + \zeta\wedge\varepsilon\mathbb{1}_{\{\theta_n\leq t\}\cap\{\theta_n<\tau_n^r\}}\ . \end{aligned}$$

Therefore, from (1.4.47) and the previous inequality, we have

$$-\zeta\wedge\varepsilon\mathbb{1}_{\{\theta_n>t\}\cup\{\theta_n=\tau_n^r\}} \leq -\zeta\wedge\varepsilon + Y_{t\wedge\theta_n}^{n,i_0} - \varphi_{i_0}\left(t\wedge\theta_n, X_{t\wedge\theta_n}^{n,i_0}\right)\ .$$

Applying the dynamics (1.2.10) of $\hat{Z}^{t_n,\delta_{(i_0,\hat{x}_n)},\alpha^n}$ to the function $(t,x,y,i)\mapsto y-\varphi_i(t,x)$, it follows from the definition of ψ_n, the one of θ_n, and (1.4.45) that

$$\begin{aligned} -\zeta\wedge\varepsilon\mathbb{1}_{\{\theta_n>t\}\cup\{\theta_n=\tau_n^r\}} \quad &\leq \quad \beta_n - \zeta\wedge\varepsilon + \int_{t_n}^{t\wedge\theta_n} \psi_s^{n\top}\mathrm{d}B_u^{i_0} \\ &+ \quad \int_{t_n}^{t\wedge\theta_n}\Big[-\partial_t\varphi_{i_0}\left(u,X_u^{n,i_0}\right) + \lambda_Y\left(X_u^{n,i_0},Y_u^{n,i_0},\alpha_u^{n,i_0}\right) - L^{\alpha_u^{n,i_0}}\varphi_{i_0}\left(u,X_u^{n,i_0}\right)\Big]\mathrm{d}u \\ &+ \quad \int_{(t_n,\theta_n\wedge t]}\sum_{k\geq 0}\left((k-1)Y_u^{n,i_0} - \left(\sum_{\ell=0}^{k-1}\varphi_{i_0\ell}-\varphi_{i_0}\right)\left(u,X_u^{n,i_0}\right)\right)Q^{i_0}(\mathrm{d}u\mathrm{d}k) \\ &\leq \quad \beta_n - \zeta\wedge\varepsilon + \int_{t_n}^{t\wedge\theta_n}\psi_s^{n\top}\mathrm{d}B_u^{i_0} \\ &+ \quad \int_{t_n}^{t\wedge\theta_n}\Big[-\partial_t\varphi_{i_0}\left(u,X_u^{n,i_0}\right) + \lambda_Y\left(X_u^{n,i_0},Y_u^{n,i_0},\alpha_u^{n,i_0}\right) - L^{\alpha_u^{n,i_0}}\varphi_{i_0}\left(u,X_u^{n,i_0}\right)\Big]\mathbb{1}_{A_n}(u)\mathrm{d}u \\ &+ \quad \int_{(t_n,\theta_n\wedge t]}\sum_{k\geq 0}\left((k-1)Y_u^{n,i_0} - \left(\sum_{\ell=0}^{k-1}\varphi_{i_0\ell}-\varphi_{i_0}\right)\left(u,X_u^{n,i_0}\right)\right)Q^{i_0}(\mathrm{d}u\mathrm{d}k)\ . \end{aligned}$$

We, then, get

$$-\zeta\wedge\varepsilon\mathbb{1}_{\{\theta_n>t\}\cup\{\theta_n=\tau_n^r\}} \quad \leq \quad M_{t\wedge\theta_n}^{B,n} + M_{t\wedge\theta_n}^{Q,n}\ , \tag{1.4.48}$$

where

$$M_s^{B,n} \;:=\; \beta_n - \zeta\wedge\varepsilon + \int_{t_n}^s b_u^n \mathrm{d}u + \int_{t_n}^s \psi_s^{n\top}\mathrm{d}B_u^{i_0} ,$$
$$\text{with } b_s^n := \left[-\partial_t\varphi_{i_0}\left(s,X_s^{n,i_0}\right) + \lambda_Y\left(X_s^{n,i_0},Y_s^{n,i_0},\alpha_s^{n,i_0}\right) - L^{\alpha_s^{n,i_0}}\varphi_{i_0}\left(s,X_s^{n,i_0}\right)\right]\mathbb{1}_{A_n}(s) ,$$
$$M_s^{Q,n} \;:=\; \int_{(t_n,s]}\sum_{k\geq 0} d_s^n(k)Q^{i_0}(\mathrm{d}u\mathrm{d}k) , \quad \text{with } d_s^n(k) := (k-1)Y_u^{n,i_0} - \left(\sum_{\ell=0}^{k-1}\varphi_{i_0\ell} - \varphi_{i_0}\right)(u,X_u^{n,i_0}) ,$$

for $s\in[t_n,T]$. Therefore, let now $L^{n,m}$ be the exponential local martingale defined by $L_{t_n}^{n,m}=1$ and

$$\mathrm{d}L_s^{n,m} = L_{s-}^{n,m}\left(-b_s^n|\psi_s^n|^{-2}\psi_s^{n\top}\mathrm{d}B_s^{i_0} + \sum_{k\geq 0}\left(\frac{1}{m}-1\right)\left(Q^{i_0}(\mathrm{d}u\mathrm{d}k) - \gamma p_k\delta_k(\mathrm{d}k)\mathrm{d}u\right)\right) ,$$

for $s\in[t_n,T]$. $L^{n,m}$ is well defined by (1.4.46), Assumption A1, and the definition of the set of admissible controls $\mathcal{A}$ and it is a martingale, from the definition of θ_n. Using now Girsanov theorem for jump-diffusion processes (see, *e.g.*, Theorem 1.35 in [129]) and the definition of θ_n, we get that $L_{\cdot\wedge\theta_n}^{n,m}M_{\cdot\wedge\theta_n}^{B,n} + L_{\cdot\wedge\theta_n}^{n,m}\left(M_{\cdot\wedge\theta_n}^{Q,n} - D_{\cdot\wedge\theta_n}^{n,m}\right)$ is a martingale, with

$$D_s^{n,m} \;:=\; \frac{\gamma}{m}\int_{t_n}^s\sum_{k\geq 0} d_u^n(k)p_k\mathrm{d}u .$$

Combining this result with (1.4.48), we get

$$\begin{aligned}-\zeta\wedge\varepsilon\mathbb{E}[\mathbb{1}_{\{\theta_n=\tau_n^r\}}L_{\theta_n}^{n,m}] \;&\leq\; \mathbb{E}\left[L_{\theta_n}^{n,m}M_{\theta_n}^{B,n} + L_{\theta_n}^{n,m}\left(M_{\theta_n}^{Q,n} - D_{\theta_n}^{n,m}\right)\right] + \mathbb{E}\left[L_{\theta_n}^{n,m}D_{\theta_n}^{n,m}\right]\\ &\leq\; L_{t_n}^{n,m}M_{t_n}^{B,n} + L_{t_n}^{n,m}M_{t_n}^{Q,n} + \mathbb{E}\left[L_{\theta_n}^{n,m}D_{\theta_n}^{n,m}\right] = \beta_n - \zeta\wedge\varepsilon + \mathbb{E}\left[L_{\theta_n}^{n,m}D_{\theta_n}^{n,m}\right] .\end{aligned}$$

Since $L_{\cdot\wedge\theta_n}^{n,m}$ is a martingale and θ_n is a stopping time bounded by ε, we have $\mathbb{E}[L_{\theta_n}^{n,m}] = L_{t_n}^n = 1$. Moreover, (1.4.39) and Assumption A1 imply

$$|D_s^{n,m}(k)| \leq \frac{C_\varepsilon}{m}, \quad \text{with } C_\varepsilon := \gamma TM\left(\varepsilon + 2\left(\sup_{B_\varepsilon(t_0,x_0)}\varphi\right)\right) ,$$

for $s\in[t_n,\theta_n]$. Therefore, the previous inequality becomes

$$\zeta\wedge\varepsilon\mathbb{E}\left[\mathbb{1}_{\{\theta_n<\tau_n^r\}}L_{\theta_n}^n\right] \;\leq\; \beta_n + \frac{C_\varepsilon}{m} . \tag{1.4.49}$$

We next define the probability measure on $\mathcal{F}_T^{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{P}^n}{d\mathbb{P}}\right|_{\mathcal{F}_T^{\mathbb{P}}} \;=\; L_{\theta_n}^{n,m}$$

and denote by $\mathbb{E}^n$ the expectation under $\mathbb{P}^n$. Using Girsanov theorem, we notice that τ_n^r under $\mathbb{P}^n$ is distributed as an exponential random variable with parameter γ/m. This entails that

$$\mathbb{E}\left[\mathbb{1}_{\{\theta_n<\tau_n^r\}}L_{\theta_n}^{n,m}\right] \;\geq\; \mathbb{E}\left[\mathbb{1}_{\{\tau_n^r>\varepsilon\}}L_{\theta_n}^{n,m}\right] \;=\; \mathbb{E}^n\left[\mathbb{1}_{\{\tau_n^r>\varepsilon\}}\right] \;=\; \exp(-\varepsilon\gamma/m) .$$

Comparing with (1.4.49), we have $0 \leq \beta_n - \zeta \wedge \varepsilon \ \exp(-\varepsilon\gamma/m) + C_\varepsilon/m$, which contradicts (1.4.44) for n and m large enough.

Viscosity subsolution property

Step 1. Let $\varphi \in C^0([0,T]\times\mathbb{R}^d)$, $\varphi_i \in C^{1,2}([0,T]\times\mathbb{R}^d)$, for $i \in \mathcal{I}$, and $(t_0, x_0, i_0) \in [0,T)\times\mathbb{R}^d\times\mathcal{I}$, be such that

$$\sup_i |\varphi_i| \quad \leq \quad \varphi$$

and

$$0 = \left(\bar{v}_{i_0}^* - \varphi_{i_0}\right)(t_0, x_0) = \max_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}} \left(\bar{v}_i^* - \varphi_i\right)(t,x) \ . \tag{1.4.50}$$

Without loss of generality, we can assume that the maximum is strict in (t,x) once fixed i_0. Assume, by contradiction, that

$$4\eta \ = \ \min\left\{ \left(-\partial_t\varphi_{i_0} + F_*\varphi_{i_0}\right)(t_0,x_0) \ ; \ \left(\varphi_{i_0} - \sup_{0\leq k<\bar{K}} \varphi_{i_0 k}\right)(t_0,x_0) \right\} > 0 \ . \tag{1.4.51}$$

By (1.4.34), Assumption A3 and (1.4.51) we may find $\varepsilon > 0$ such that

$$\rho(t,x,y) = -\partial_t\varphi_{i_0}(t,x) + \lambda_Y(x,y,\hat{a}(x,D\varphi_{i_0}(t,x))) - L^{\hat{a}(x,D\varphi_{i_0}(t,x))}\varphi_{i_0}(t,x) \quad \geq \quad \eta, \tag{1.4.52}$$

$$\left(\varphi_{i_0} - \sup_{0\leq k<\bar{K}} \varphi_{i_0 k}\right)(t,x) \quad \geq \quad \eta, \tag{1.4.53}$$

for all $(t,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}$ such that $(t,x) \in B_\varepsilon(t_0,x_0)$ and $|y - \varphi_{i_0}(t,x)| \leq \varepsilon$, where $\hat{a}$ is a locally Lipschitz map satisfying

$$\hat{a}(x, D\varphi_{i_0}(t,x)) \in \mathcal{N}_0(x, \varphi_{i_0}(t,x), D\varphi_{i_0}(t,x)) \text{ on } B_\varepsilon(t_0,x_0) \ . \tag{1.4.54}$$

Moreover, since (t_0,x_0) is a strict maximizer, we have

$$-\zeta \quad = \quad \max_{\partial_p B_\varepsilon(t_0,x_0)} (\bar{v}_{i_0}^* - \varphi_{i_0})(t,x) \quad < \quad 0 \ , \tag{1.4.55}$$

where $\partial_p B_\varepsilon(t_0,x_0) := \{t_0+\varepsilon\} \times \mathrm{cl}(B_\varepsilon(t_0,x_0)) \cup [t_0, t_0+\varepsilon) \times \partial B_\varepsilon(t_0,x_0)$ denotes the parabolic boundary of $B_\varepsilon(t_0,x_0)$.

Step 2. We now show that (1.4.52), (1.4.53), (1.4.54) and (1.4.55) lead to a contradiction to the DPP (1.3.29). Let $(t_n, x_n)_{n\geq 1}$ be a sequence such that

$$(t_n,x_n) \to (t_0,x_0) \ \text{ and } \bar{v}_{i_0}(t_n,x_n) \to \bar{v}_{i_0}^*(t_0,x_0) \quad \text{ as } \quad n \to +\infty \ .$$

We set y_0, $\hat{x}_0$, y_n, and $\hat{x}_n$ as in the proof of the viscosity supersolution property and notice that

$$\beta_n := y_n - \varphi_{i_0}(t_n,x_n) \quad \xrightarrow[n\to+\infty]{} \quad 0 \ . \tag{1.4.56}$$

Define now the following stopping times

$$\begin{aligned}
\tau_n &:= \inf\{s \geq t_n \;:\; \exists i \in \mathcal{V}^n_s\,,\ \left(s, X^{n,i}_s\right) \notin B_\varepsilon(t_0,x_0)\}\,,\\
\tau^\varepsilon_n &:= \inf\{s \geq t_n \;:\; \exists i \in \mathcal{V}^n_s\,,\ |Y^{n,i}_s - \varphi_i\left(s, X^{n,i}_s\right)| \geq \varepsilon\}\,,\\
\tau^r_n &:= \inf\{s \geq t_n \;:\; Q^{i_0}((t_n,s]\times\mathbb{N}) \geq 1\ \}\,,\\
\theta_n &:= \tau_n \wedge \tau^\varepsilon_n \wedge \tau^r_n\,.
\end{aligned}$$

To alleviate the notations, we shall write

$$X^{n,i}_\cdot := X^{t_n,\delta_{(i_0,x_n)},\alpha^n,i}_\cdot\,,\ Y^{n,i}_\cdot := Y^{t_n,\delta_{(i_0,x_n,y_n)},\alpha^n,i}_\cdot\,,\ \hat{X}^{n,i}_\cdot := \left(X^{n,i}_\cdot, Y^{n,i}_\cdot\right)\,,$$
$$\hat{Z}^n_\cdot := \hat{Z}^{t_n,\delta_{(i_0,\hat{x}_n)},\hat{\alpha}^n}_\cdot\,,\quad \text{and}\quad \mathcal{V}^n_\cdot := \mathcal{V}^{t_n,\delta_{(i_0,x_n)},\hat{\alpha}^n}_\cdot\,,$$

where $\hat{\alpha}^n$ is the feedback control process given by $\hat{\alpha}^{n,i}_\cdot := \hat{a}(X^{n,i}_\cdot, D\varphi_{i_0}(\cdot, X^{n,i}_\cdot))$ defined on $[t_n,\theta_n)$ for $n \geq 1$. Since $\hat{a}$ is locally Lipschitz, this solution is well-defined. Since $\bar{v}_i \leq \bar{v}^*_i \leq \varphi_i$, we deduce from (1.4.55) and the definition of θ_n that on $\{\theta_n < \tau^r_n\}$ we have

$$\begin{aligned}
Y^{n,i_0}_{\theta_n} - \bar{v}_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right) &\geq \mathbb{1}_{\{\theta_n=\tau^\varepsilon_n\}}\left(Y^{n,i_0}_{\theta_n} - \varphi_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right)\right)\\
&\quad +\mathbb{1}_{\{\theta_n=\tau_n\}}\left(Y^{n,i_0}_{\theta_n} - \bar{v}^*_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right)\right)\\
&= \varepsilon\mathbb{1}_{\{\theta_n=\tau^\varepsilon_n\}} + \mathbb{1}_{\{\theta_n=\tau_n<\tau^\varepsilon_n\}}\left(Y^{n,i_0}_{\theta_n} - \bar{v}^*_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right)\right)\\
&\geq \varepsilon\mathbb{1}_{\{\theta_n=\tau^\varepsilon_n\}} + \mathbb{1}_{\{\theta_n=\tau_n<\tau^\varepsilon_n\}}\left(Y^{n,i_0}_{\theta_n} + \zeta - \varphi_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right)\right)\\
&\geq \varepsilon\wedge\zeta + \mathbb{1}_{\{\theta_n=\tau_n<\tau^\varepsilon_n\}}\left(Y^{n,i_0}_{\theta_n} - \varphi_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right)\right)\,.
\end{aligned}$$

Secondly, on $\{\theta_n = \tau^r_n\}$, using the continuity of the trajectories of the particles $Y^{i_0\ell}_{\theta_n} = Y^{i_0}_{\theta_n}$ and $X^{i_0\ell}_{\theta_n} = X^{i_0}_{\theta_n}$ for all $i_0\ell \in \mathcal{V}^n_{\theta_n}$, we have

$$Y^{n,i_0\ell}_{\theta_n} - \varphi_{i_0\ell}\left(\theta_n, X^{n,i_0\ell}_{\theta_n}\right) = Y^{n,i_0}_{\tau^r_n} - \varphi_{i_0}\left(\tau^r_n, X^{n,i_0}_{\tau^r_n}\right) + \varphi_{i_0}\left(\tau^r_n, X^{n,i_0}_{\tau^r_n}\right) - \varphi_{i_0\ell}\left(\tau^r_n, X^{n,i_0}_{\tau^r_n}\right)\,,$$

and from (1.4.53),

$$Y^{n,i_0\ell}_{\theta_n} - \varphi_{i_0\ell}\left(\theta_n, X^{n,i_0\ell}_{\theta_n}\right) \geq Y^{n,i_0}_{\theta_n} - \varphi_{i_0}\left(\theta_n, X^{n,i_0}_{\theta_n}\right) + \eta\,, \tag{1.4.57}$$

for all $i_0\ell \in \mathcal{V}^n_{\theta_n}$.

From (1.4.52) and (1.4.54), we get by Itô's formula

$$Y^{n,i}_{\theta_n} - \bar{v}_i\left(\theta_n, X^{n,i}_{\theta_n}\right) \geq \varepsilon\wedge\zeta\wedge\eta + \beta_n\,,\qquad \text{for } i \in \mathcal{V}^n_{\theta_n}\,.$$

Since $y_n = \bar{v}_{i_0}(t_n,x_n) - n^{-1} < \bar{v}_{i_0}(t_n,x_n)$, this is in contradiction with the DPP (1.3.29) for n large enough by (1.4.56).

1.4.4 Viscosity properties on $\{T\} \times \mathbb{R}^d \times \mathcal{I}$

Viscosity supersolution

Fix $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{I}$ and $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$ satisfying

$$0 \;=\; \bar{v}_{i_0,*}(T, x_0) - \varphi_{i_0}(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d} \left(\bar{v}_{\cdot,*}(T,\cdot) - \varphi_{\cdot}\right) .$$

Without loss of generality, we can take this minimum to be strict in x once fixed i_0.

Step 1. From the convention $\sup\emptyset := -\infty$ and since $\bar{v}$ is a viscosity supersolution for (1.4.33) on $[0,T) \times \mathbb{R}^d \times \mathcal{I}$, we have

$$\delta^*\bar{v}_{\cdot,*} \geq 0 \;\text{ on } [0,T) \times \mathbb{R}^d \times \mathcal{I} ,$$

in the viscosity sense. From the upper-semicontinuity of δ^*, we can, then, deduce by a standard argument (see, *e.g.*, proof of Lemma 5.2 in [146]) that $\delta^*\varphi(x_0) \geq 0$.

Step 2. We now prove

$$\varphi_{i_0}(x_0) - \sup_{0\leq k<\bar{K}} \varphi_{i_0 k}(x_0) \;\geq\; 0 .$$

From the definition of $\bar{v}_*$, there exists a sequence $(s_n, \xi_n)_{n\geq 1}$ converging to (T, x_0) such that $s_n < T$ for $n \geq 1$ and

$$\lim_{n\to\infty} \bar{v}_{i_0,*}(s_n, \xi_n) = \bar{v}_{i_0,*}(T, x_0) .$$

For $n \geq 1$, consider the auxiliary test function

$$\varphi_{n,i}(t,x) := \varphi_i(x) - \frac{1}{2}|x - x_0|^2 + \frac{T-t}{(T-s_n)^2} , \qquad \text{for } (t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I} .$$

Let $B_1(x_0)$ be the unit open ball in $\mathbb{R}^d$ centered at x_0 and choose $(t_n, x_n) \in [s_n, T] \times \bar{B}_1(x_0)$, which minimizes the difference $\bar{v}_{i_0,*} - \varphi_{n,i_0}$ on $[s_n, T] \times \bar{B}_1(x_0)$. We claim that, for n large enough $t_n < T$, and x_n converges to x_0. Indeed, we first have

$$(\bar{v}_{i_0,*} - \varphi_{n,i_0})(s_n, x_0) = (\bar{v}_{i_0,*} - \varphi_{i_0})(s_n, x_0) - \frac{1}{(T-s_n)} .$$

Since $(\bar{v}_{i_0,*} - \varphi_{i_0})(s_n, x_0)$ is bounded and $s_n \to T-$, for sufficiently large n, we have

$$(\bar{v}_{i_0,*} - \varphi_{n,i_0})(s_n, x_0) \;<\; 0 .$$

On the other hand, for any $x \in \bar{B}_1(x_0)$

$$(\bar{v}_{i_0,*} - \varphi_{n,i_0})(T, x) \;=\; \bar{v}_{i_0,*}(T,x) - \varphi_{i_0}(x) + \frac{1}{2}|x - x_0|^2 \;\geq\; \bar{v}_{i_0,*}(T,x) - \varphi_{i_0}(x) \;\geq\; 0 .$$

Comparing the two inequalities, we conclude that $t_n < T$ for large n. Let x^* be an adherence value of the sequence $(x_n)_{n\geq 1}$. Since $t_n \geq s_n$ and (t_n, x_n) minimizes the difference $(\bar{v}_{i_0,*} - \varphi_{n,i_0})$, we have

$$(\bar{v}_{i_0,*}(T,.)-\varphi_{i_0})(x^*)-(\bar{v}_{i_0,*}(T,.)-\varphi_{i_0})(x_0)$$
$$\leq \liminf_{n\to\infty}(\bar{v}_{i_0,*}-\varphi_{n,i_0})(t_n,x_n)-(\bar{v}_{i_0,*}-\varphi_{n,i_0})(s_n,\xi_n)-\frac{1}{2}|x_n-x_0|^2 \leq -\frac{1}{2}|x^*-x_0|^2 .$$

Since x_0 minimizes the difference $\bar{v}_{i_0,*}(T,\cdot)-\varphi_{i_0}$, we have

$$0 \quad \leq \quad (\bar{v}_{i_0,*}(T,\cdot)-\varphi_{i_0})(x^*)-(\bar{v}_{i_0,*}(T,\cdot)-\varphi_{i_0})(x_0) \;\leq\; -\frac{1}{2}|x^*-x_0|^2 .$$

Hence, $x^*=x_0$ and $(x_n)_{n\geq 1}$ converges to x_0.

We now use the viscosity supersolution property of $\bar{v}$ on $[0,T)\times\mathbb{R}^d\times\mathcal{I}$ with the test function $\tilde{\varphi}_{n,\cdot}=\varphi_{n,\cdot}+\bar{v}_{*,i_0}(t_n,x_n)-\varphi_{n,i_0}(t_n,x_n)$ and get

$$\tilde{\varphi}_{n,i_0}(t_n,x_n)-\sup_{0\leq k<\bar{K}}\tilde{\varphi}_{n,i_0k}(t_n,x_n) \quad \geq \quad 0 , \tag{1.4.58}$$

for all $n\geq 1$. This entails

$$\varphi_{i_0}(x_n)-\sup_{0\leq k<\bar{K}}\varphi_{i_0k}(x_n)=\tilde{\varphi}_{n,i_0}(t_n,x_n)-\sup_{0\leq k<\bar{K}}\tilde{\varphi}_{n,i_0k}(t_n,x_n) .$$

Finally, since x_n converges to x_0, by sending n to infinity, we obtain $\varphi_{i_0}(x_0)-\sup_{0\leq k<\bar{K}}\varphi_{i_0k}(x_0)\geq 0$.

Step 3. We now prove the last assertion. Assume that

$$F^*\varphi_{i_0}(x_0)<\infty \quad \text{and} \quad \varphi_{i_0}(x_0)=\bar{v}_{i_0,*}(T,x_0)<g_{i_0}$$

and let us work towards a contradiction. Since $\bar{v}_\cdot(T,\cdot)=g_\cdot$ by the definition of the problem, there is a constant $\eta>0$ such that

$$\varphi_{i_0}-\bar{v}_{i_0}(T,\cdot) \quad = \quad \varphi_{i_0}-g_{i_0} \;\leq\; -\eta \;\text{ on } B_\varepsilon(x_0) ,$$

for some $\varepsilon>0$. Since x_0 is a strict minimizer, let ζ be

$$2\zeta=\min_{x\in\partial B_\varepsilon(x_0)}\bar{v}_{i_0,*}(T,x)-\varphi_{i_0}(x)>0 .$$

It follows that there exists $r>0$ such that $\bar{v}_{i_0}(t,x)-\varphi_{i_0}(x)\geq\zeta>0$ for all $(t,x)\in[T-r,T]\times\partial B_\varepsilon(x_0)$. This holds, otherwise, for each $r>0$, we could find $(t_r,x_r)\in[T-r,T]\times\partial B_\varepsilon(x_0)$ such that $\bar{v}_{i_0}(t_r,x_r)-\varphi_{i_0}(x_r)\leq\zeta$. Sending r to 0, since $\partial B_\varepsilon(x_0)$ is compact, up to a subsequence we would have $\bar{v}_{i_0,*}(T,x^*)-\varphi_{i_0}(x^*)\leq\zeta$ for some $x^*\in\partial B_\varepsilon(x_0)$, in contradiction with the definition of ζ.

Therefore, we have

$$\bar{v}_{i_0}(t,x)-\varphi_{i_0}(x)\geq\zeta\wedge\eta>0, \qquad \text{for } (t,x)\in\Big([T-r,T]\times\partial B_\varepsilon(x_0)\Big)\cup\Big(\{T\}\times B_\varepsilon(x_0)\Big) .$$

Since $F^*\varphi_{i_0}(x_0)<\infty$, up to smaller $\varepsilon>0$ we have

$$\lambda_Y(x,y,a)-L^a\varphi_{i_0}(x)\leq C , \qquad \text{for all } a\in\mathcal{N}_\varepsilon(x,\varphi_{i_0}(x),D\varphi_{i_0}(x)) \text{ and } (x,y)\in\mathbb{R}^d\times\mathbb{R}$$
$$\text{such that } x\in B_\varepsilon(x_0) \text{ and } |y-\varphi_{i_0}(x)|\leq\varepsilon ,$$

for some constant $C > 0$. Consider $\tilde{\varphi}_i(t,x) := \varphi_i(x) + 2C(t-T)$. Then, for sufficiently small $r > 0$,

$$\bar{v}_{i_0}(t,x) - \tilde{\varphi}_{i_0}(t,x) \geq \frac{1}{2}(\zeta \wedge \eta) \quad > \quad 0 \; ,$$

for $(t,x) \in ([T-r,T] \times \partial B_\varepsilon(x_0)) \cup (\{T\} \times B_\varepsilon(x_0))$, and

$$-\partial_t \tilde{\varphi}_{i_0}(t,x) + \lambda_Y(x,y,a) - L^a \tilde{\varphi}_{i_0}(t,x) \quad \leq \quad -C \; ,$$

for all $a \in \mathcal{N}_\varepsilon(x, \tilde{\varphi}_{i_0}(t,x), D\tilde{\varphi}_{i_0}(t,x))$ and $(x,y) \in \mathbb{R}^d \times \mathbb{R}$ such that $x \in B_\varepsilon(x_0)$ and $|y - \tilde{\varphi}_{i_0}(t,x)| \leq \varepsilon$. By following the same arguments as in Step 3 of Section 1.4.3, the latter inequalities lead to a contradiction of the DPP (1.3.29).

Viscosity subsolution

Fix $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{I}$ and $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$ satisfying

$$0 \;=\; \bar{v}^*_{i_0}(T,x_0) - \varphi_{i_0}(x_0) \quad = \quad \max_{\mathcal{I} \times \mathbb{R}^d} (\bar{v}^*_{\cdot}(T,\cdot) - \varphi_{\cdot}) \; .$$

Without loss of generality, we can take this maximum to be strict in x once have fixed i_0. Assume, by contradiction, that $\delta_* \varphi_{i_0}(x_0) > 0$ and

$$4\eta \;=\; \min \left\{ \varphi_{i_0}(x_0) - g_{i_0}(x_0) \; ; \; \left(\varphi_{i_0} - \sup_{0 \leq k < \bar{K}} \varphi_{i_0 k} \right)(x_0) \right\} > 0 \quad . \tag{1.4.59}$$

By (1.4.37) and Assumption A3, we can find $r > 0$ and a locally Lipschitz map $\hat{a}$ satisfying

$$\hat{a}(x, D\varphi_{i_0}(x)) \in \mathcal{N}_0(x, \varphi_{i_0}(x), D\varphi_{i_0}(x)) \; , \tag{1.4.60}$$

for all $x \in B_r(x_0)$. Set $\tilde{\varphi}_i(t,x) := \varphi_i(x) + \sqrt{T-t}$. Since $\partial_t \tilde{\varphi}_i(t,x) \to -\infty$ as $t \to T$, for $r, \varepsilon > 0$ small enough we get

$$\rho(t,x,y) = -\partial_t \tilde{\varphi}_{i_0}(t,x) + \lambda_Y\left(x,y,\hat{a}\left(x, D\tilde{\varphi}_{i_0}(t,x)\right)\right) - L^{\hat{a}(x,D\tilde{\varphi}_{i_0}(t,x))} \tilde{\varphi}_{i_0}(t,x) \geq \eta \; , \tag{1.4.61}$$

for all $(t,x,y) \in [T-r,T) \times \mathbb{R}^d \times \mathbb{R}$ such that $x \in B_r(x_0)$ and $|y - \tilde{\varphi}_{i_0}(t,x)| \leq \varepsilon$. Combining

$$\left(\tilde{\varphi}_{i_0} - \sup_{0 \leq k < \bar{K}} \tilde{\varphi}_{i_0 k} \right)(t,x_0) = \left(\varphi_{i_0} - \sup_{0 \leq k < \bar{K}} \varphi_{i_0 k} \right)(x_0) \; ,$$

with (1.4.59), we get

$$\left(\tilde{\varphi}_{i_0} - \sup_{0 \leq k < \bar{K}} \tilde{\varphi}_{i_0 k} \right)(t,x) \geq \eta \; , \qquad \text{for all } (t,x) \in [T-r,T] \times B_r(x_0) \; , \tag{1.4.62}$$

for $r > 0$ small enough.

Since $\bar{v}^*_{i_0} - \tilde{\varphi}_{i_0}$ is upper-semicontinuous and $\left(\bar{v}^*_{i_0} - \tilde{\varphi}_{i_0}\right)(T,x_0) = 0$, we have

$$\bar{v}^*_{i_0}(t,x) \leq \tilde{\varphi}_{i_0}(t,x) + \varepsilon/2 \; , \qquad \text{for all } (t,x) \in [T-r,T] \times B_r(x_0) \tag{1.4.63}$$

and, from $\bar{v}_{\cdot}(T,\cdot) = g_{\cdot}$,

$$\tilde{\varphi}_{i_0} - \bar{v}_{i_0}(T,\cdot) = \tilde{\varphi}_{i_0} - g_{i_0} \geq \eta \;\text{ on } B_r(x_0) ,$$

for r small enough. Since x_0 is a strict maximizer for $v_{i_0,*}(T,\cdot) - \varphi_{i_0}$, we can define $\zeta > 0$ such that

$$-2\zeta = \max_{x\in\partial B_r(x_0)} \bar{v}^*_{i_0}(T,x) - \varphi_{i_0}(x) < 0 .$$

It follows that $\bar{v}_{i_0}(t,x) - \tilde{\varphi}_{i_0}(x) \leq -\zeta < 0$ for all $(t,x) \in [T-r,T] \times \partial B_r(x_0)$, for $r > 0$ small enough. This means

$$\bar{v}_{i_0}(t,x) - \tilde{\varphi}_{i_0}(x) \leq -\zeta \wedge \eta \;\text{ for all } (t,x) \in ([T-r',T] \times \partial B_r(x_0)) \cup (\{T\} \times B_r(x_0)) \;. \tag{1.4.64}$$

Finally, following the arguments in Step 2 of Section 1.4.3, we see that (1.4.60), (1.4.61), (1.4.62), (1.4.63), (1.4.64), lead to a contradiction of (1.3.29).

1.4.5 Uniqueness

We turn to the uniqueness of the solution to the dynamic programming equation (1.4.33)-(1.4.38). To this end, we need to introduce additional assumptions. We first recall that the Hausdorff distance $d_{\mathcal{H}}$ on closed subsets of A is defined by

$$d_{\mathcal{H}}(B,C) \;:=\; \min\{r \geq 0 \;:\; B \subseteq C_r \text{ and } C \subseteq B_r\} ,$$

for $B, C \subseteq A$ closed and nonempty, with

$$D_r \;=\; \{a \in A \;:\; \exists a' \in D ,\; d_A(a,a') \leq r\} , \tag{1.4.65}$$

for any $D \subseteq A$ and any $r \geq 0$. We use the convention

$$d_{\mathcal{H}}(B,C) \;=\; +\infty ,$$

if $B = \emptyset$ or $C = \emptyset$.

Assumption A4. *(i) The functions λ and σ do not depend on the control, i.e., $\lambda : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$. The function σ_Y does not depend on the variable y, i.e., $\sigma_Y : \mathbb{R}^d \times A \to \mathbb{R}^d$. Therefore, also $\mathcal{N}$ does not depend on y.*

(ii) There exist two constants $C > 0$ and $\eta \in (0,1]$ such that the function w appearing in Assumption A1(iv) satisfies $w(x) \leq Cx^\eta$ for $x \in \mathbb{R}_+$.

(iii) There exists a constant $C > 0$ such that

$$d_{\mathcal{H}}\left(\, \mathcal{N}_\varepsilon(x,p) \,,\, \mathcal{N}_{\varepsilon'}(x',p') \,\right) \leq C\left(|p-p'| + \varepsilon + \varepsilon'\right)(1+|x|) + C|x-x'| ,$$

for all $\varepsilon, \varepsilon' \geq 0$, $x, x', p, p' \in \mathbb{R}^d$.

(iv) $0 \in \mathrm{Int}(\mathbf{N}(x,p))$ for all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$.

Remark 1.4.3. *Using the convention below* (1.4.65)*, the combination of the points (iii) and (iv) implies that $\mathcal{N}_\varepsilon(x,p) \neq \emptyset$ for any $(\varepsilon,x,p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.*

In particular, we always have that $\delta\varphi > 0$ for any $\varphi \in C^2(\mathbb{R}^d)$. Therefore, the terminal viscosity supersolution (resp. subsolution) property takes the following form

$$\min\left\{ (\varphi_i(x) - g_i(x))\,\mathbb{1}_{F^*\varphi_i(x)<\infty}\ ;\ \left(\varphi_i - \sup_{0\leq k<\bar{K}} \varphi_{ik}\right)(x) \right\} \ \geq\ 0 \qquad (1.4.66)$$

$$(resp. \qquad \min\left\{ (\varphi_i(x) - g_i(x))\ ;\ \left(\varphi_i - \sup_{0\leq k<\bar{K}} \varphi_{ik}\right)(x) \right\} \ \leq\ 0)\ , \qquad (1.4.67)$$

for $(x,i) \in \mathbb{R}^d \times \mathcal{I}$ and $(\varphi_j)_{j\in\mathcal{I}}$ a test function according to Definition 1.4.3.

Lemma 1.4.2. *Let $u:\ [0,T]\times\mathbb{R}^d\times\mathcal{I}$ be a lower semi-continuous supersolution of (1.4.33)-(1.4.66). Define the function $\Lambda:\ [0,T]\times\mathbb{R}^d \to \mathbb{R}$ by*

$$\Lambda(t,x) := \theta e^{-\kappa t}(1+|x|^{2\gamma+2})\ , \qquad \textit{for } (t,x) \in [0,T]\times\mathbb{R}^d\ ,$$

with $\theta, \kappa, \gamma \in \mathbb{R}_+$. Then, under Assumptions A1 and A4, for any $\gamma \geq 0$, there exists $\kappa_0 > 0$ such that the function $u + \Lambda$ is a supersolution to (1.4.33)-(1.4.38), for any $\kappa \geq \kappa_0$ and $\theta > 0$.

Proof. Let $\bar{\varphi} \in C^0([0,T]\times\mathbb{R}^d)$ and $\varphi_j \in C^{1,2}([0,T]\times\mathbb{R}^d)$, for $j \in \mathcal{I}$, be such that the function $\varphi_\cdot - (u+\Lambda)$ has a local maximum in (t,x,i) equal to 0 and $\sup_{j\in\mathcal{I}}|\varphi_j| \leq \bar{\varphi}$. Since u is a supersolution for (1.4.33), we have

$$\min\left\{-\partial_t(\varphi_i - \Lambda)(t,x) + F^*(\varphi_i-\Lambda)(t,x)\ ;\ \left((\varphi_i - \Lambda) - \sup_{0\leq k<\bar{K}}(\varphi_{ik} - \Lambda)\right)(t,x)\right\} \ \geq\ 0\ .$$

Clearly, we have

$$\left(\varphi_i - \sup_{0\leq k<\bar{K}} \varphi_{ik}\right)(t,x) \ =\ \left((\varphi_i - \Lambda) - \sup_{0\leq k<\bar{K}}(\varphi_{ik} - \Lambda)\right)(t,x) \ \geq\ 0\ . \qquad (1.4.68)$$

Thus, we can focus on proving

$$-\partial_t\varphi_i(t,x) + F^*\varphi_i(t,x) \ \geq\ 0\ .$$

If $F^*\varphi_i(t,x) = +\infty$, then, the inequality is obvious. Suppose now that $F^*\varphi_i(t,x) < +\infty$. From Assumption A4, we get that F^* is locally bounded and, as u is a viscosity supersolution to (1.4.33), we get

$$-\partial_t(\varphi_i - \Lambda)(t,x) + F^*(\varphi_i - \Lambda)(t,x) \ \geq\ 0\ .$$

From the definition of Λ and F, Assumption A4 and the continuity of the functions considered, we get

$$\begin{aligned}
&-\partial_t\varphi_i(t,x)-\theta\kappa e^{-\kappa t}(1+|x|^{2\gamma+2})\\
&+\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |(\varphi_i-\Lambda)(t,x)-y'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-p|\leq\varepsilon}}\sup_{a\in\mathcal{N}_\varepsilon(x',p)}\{\lambda_Y(x',a)\}\\
&-\lambda(x)^\top D\varphi_i(t,x)+\theta e^{-\kappa t}\lambda(x)^\top D|x|^{2\gamma+2}\\
&-\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)D^2\varphi_i(t,x)\right)+\theta e^{-\kappa t}\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)D^2|x|^{2\gamma+2}\right)\quad\geq\quad 0\,. \qquad (1.4.69)
\end{aligned}$$

Define the function $\Gamma_\varepsilon:\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ by $\Gamma_\varepsilon('x,y',p):=\sup_{a\in\mathcal{N}_\varepsilon(x',p)}\{\lambda_Y(x',y',a)\}$, for $(x',y',p)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$. Then, we get, from (1.4.69),

$$\begin{aligned}
&-\partial_t\varphi_i(t,x)+\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D\varphi_i(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)\\
&-\lambda(x)^\top D\varphi_i(t,x)-\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)D^2\varphi_i(t,x)\right)\quad\geq\\
&\theta\kappa e^{-\kappa t}(1+|x|^{2\gamma+2})+\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D\varphi_i(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)\\
&-\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |(\varphi_i-\Lambda)(t,x)-y'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)\\
&-\theta e^{-\kappa t}\lambda(x)^\top D|x|^{2\gamma+2}-\theta e^{-\kappa t}\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)D^2|x|^{2\gamma+2}\right)\quad=\\
&\theta\kappa e^{-\kappa t}(1+|x|^{2\gamma+2})-\theta e^{-\kappa t}\lambda(x)^\top D|x|^{2\gamma+2}\\
&-\theta e^{-\kappa t}\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)D^2|x|^{2\gamma+2}\right)+\Delta\Gamma^1(t,x)+\Delta\Gamma^2(t,x)\,, \qquad (1.4.70)
\end{aligned}$$

where

$$\begin{aligned}
\Delta\Gamma^1(t,x)\;&:=\;\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D\varphi_i(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)-\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)\,,\\
\Delta\Gamma^2(t,x)\;&:=\;\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)-\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |(\varphi_i-\Lambda)(t,x)-y'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-p|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)\,.
\end{aligned}$$

From Assumptions A1 and A4, there exists a constant $C_1>0$ that does not depend on (t,x,i) such that

$$\begin{aligned}
\Delta\Gamma^1(t,x)\quad&\geq\quad-\lim_{\varepsilon\to 0}\sup_{\substack{|x-x'|\leq\varepsilon\\ |\varphi_i(t,x)-y'|\leq\varepsilon\\ |D\varphi_i(t,x)-p|\leq\varepsilon}\ \substack{|x-\tilde{x}'|\leq\varepsilon\\ |\varphi_i(t,x)-\tilde{y}'|\leq\varepsilon\\ |D(\varphi_i-\Lambda)(t,x)-\tilde{p}|\leq\varepsilon}}\Gamma_\varepsilon(x',y',p)-\Gamma_\varepsilon(\tilde{x}',\tilde{y}',\tilde{p})\\
&\geq\quad-C_1|D\Lambda(t,x)|^\eta(1+|x|^\eta)\,.
\end{aligned}$$

Analogously, for the second term, there exists a constant $C_2>0$ that does not depend on (t,x,i)

such that

$$\Delta\Gamma^2(t,x) \quad \geq \quad -C_2\Lambda(t,x) \; .$$

This means that, if we consider the right-hand side of (1.4.70) and the growth condition of the different terms, there exists a constant κ_0, which does not depend on θ, such that if $\kappa \geq \kappa_0$ this expression is non-negative. Henceforth, with (1.4.68), we obtain that $u + \Lambda$ is a viscosity supersolution to (1.4.33).

Finally, take $(i,x) \in \mathcal{I}\times\mathbb{R}^d$, $\varphi_j \in C^2(\mathbb{R}^d)$, for $j \in \mathcal{I}$, and $\bar\varphi \in C^0(\mathbb{R}^d)$ such that $\sup_{i\in\mathcal{I}} |\varphi_i| \leq \bar\varphi$ and

$$0 \;=\; u_{i_0,*}(T,x) + \Lambda(T,x) - \varphi_i(x) \quad = \quad \max_{\mathcal{I}\times\mathbb{R}^d}(u_{\cdot,*}(T,.) + \Lambda(T,.) - \varphi_\cdot) \; .$$

Since u is a supersolution to (1.4.38), we have

$$\varphi_i(x) - \Lambda(T,x) \quad \geq \quad g_i(x)$$

and $\varphi_i(T,x) \geq g_i(x)$ as $\Lambda \geq 0$. Combining it with (1.4.68), we obtain from Remark 1.4.3 that $u + \Lambda$ is a viscosity supersolution to (1.4.38). □

We now turn to the main result of this section, the comparison theorem. We recall that the definition of $|.|$ on $\mathcal{I}$ is given in Section 1.2.1.

Theorem 1.4.6. *Let $\bar w_\cdot$ (resp. $\bar u_\cdot$) be a lsc (resp. usc) viscosity supersolution (resp. subsolution) to* (1.4.33)-(1.4.66). *Suppose that there exists $\gamma > 0$ such that*

$$\sup_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}} \frac{|\bar w_i(t,x)| + |\bar u_i(t,x)|}{1+|x|^\gamma} < +\infty \quad , \tag{1.4.71}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\bar w_i(t,x)| + |\bar u_i(t,x)| \xrightarrow[|i|\to\infty]{} 0 \; . \tag{1.4.72}$$

Then, under Assumption A1-A2-A4, we have $\bar u_\cdot \leq \bar w_\cdot$ on $[0,T]\times\mathbb{R}^d\times\mathcal{I}$.

Proof. We proceed in six steps.

Step 1. Define $\Lambda_{\theta,\kappa}(t,x) := \theta e^{-\kappa t}(1+|x|^{2\gamma+2})$ for $(t,x) \in [0,T]\times\mathbb{R}^d$ with $\theta,\kappa \in \mathbb{R}_+$. From Lemma 1.4.2, there exist κ large enough such that for any $\theta > 0$, $\bar w_\cdot + \Lambda_{\theta,\kappa}$ is also a supersolution for (1.4.33)-(1.4.38). Set $\bar w_{i,\theta,\kappa}(t,x) := \bar w_i(t,x) + \Lambda_{\theta,\kappa}(t,x)$, for $(i,t,x) \in \mathcal{I}\times[0,T]\times\mathbb{R}^d$.

For some η, $\eta' > 0$ to be chosen below, consider $\beta_t := e^{(\eta+\eta')t}$, for $t \in [0,T]$. A straightforward derivation shows that $\beta_t \bar w_{i,\theta,\kappa}$ (resp. $\beta_t \bar u_i$) is a viscosity supersolution (resp. subsolution) to

$$\min\Big\{\eta w_i - \partial_t w_i + \tilde F(t,x,w_i,Dw_i) - \lambda^\top Dw_i - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top D^2 w_i\right) \; ; \; w_i - \sup_{0\leq k<\bar K} w_{ik}\Big\} \;=\; 0 \text{ on } [0,T)\times\mathbb{R}^d, \tag{1.4.73}$$

$$\min\Big\{w_i - \tilde g \; ; \delta w_i \; ; \; w_i - \sup_{0\leq k<\bar K} w_{ik}\Big\} \;=\; 0 \text{ on } \{T\}\times\mathbb{R}^d, \tag{1.4.74}$$

where

$$\tilde{F}(t,x,y,p) := \sup_{a\in\tilde{\mathcal{N}_0}(t,x,p)} \tilde{\lambda}_Y(x,y,a) \quad , \quad \tilde{\mathcal{N}_0}(t,x,p) := \mathcal{N}_0(x,\beta_t^{-1}p) \; ,$$
$$\tilde{\lambda}_Y(t,x,y,a) := \beta_t\lambda_Y(x,\beta_t^{-1}y,a) + \eta' y \quad , \quad \tilde{g}_i(x) := \beta_T g_i(x) \; ,$$

for all $(t,x,i,y,p,a) \in [0,T]\times\mathbb{R}^d\times\mathcal{I}\times\mathbb{R}\times\mathbb{R}^d\times A$. Since λ_Y is Lipschitz, we can choose η' large enough so that $\tilde{\lambda}_Y$ and, consequently, $\tilde{F}$ are nondecreasing in y.

Let $\varepsilon > 0$. From an analogous computation, using the monotonicity of $\tilde{F}$, we see that $\beta_t\bar{w}_{i,\theta,\kappa} + \varepsilon/2^{|i|}$ is a viscosity supersolution to

$$\eta w_i - \partial_t w_i + \tilde{F}(t,x,w_i,Dw_i) - \lambda(x)^\top Dw_i - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma(x)^\top D^2 w_i\right) \;\geq\; 0 \; , \tag{1.4.75}$$
$$\min\{w_i(T,\cdot) - \tilde{g} \; ; \; \delta w_i\} \;\geq\; 0 \; , \tag{1.4.76}$$
$$w_i - \sup_{0\le k<\bar{K}} w_{ik} \geq \frac{\varepsilon}{2^{|i|+1}} =: \Delta_i \;>\; 0 \; . \tag{1.4.77}$$

Step 2. Set $\tilde{u}_i := \beta_t\bar{u}_i$ and $\tilde{w}_{i,\theta,\kappa,\varepsilon} := \beta_t\bar{w}_i + \beta_t\Lambda_{\theta,\kappa} + \varepsilon/2^{|i|} = \beta_t\bar{w}_{i,\theta,\kappa} + \varepsilon/2^{|i|}$. To prove our result, it is enough to show that

$$\tilde{u}_i(t,x) \quad\leq\quad \tilde{w}_{i,\theta,\kappa,\varepsilon}(t,x) \; ,$$

for each $(i,t,x) \in \mathcal{I}\times[0,T]\times\mathbb{R}^d$ and $\theta,\varepsilon > 0$. Then, taking the limit as $\theta\to 0$ and $\varepsilon \to 0$, we obtain the desired result. For simplicity, we write $\tilde{w}_i$ for $\tilde{w}_{i,\theta,\kappa,\varepsilon}$ in the sequel. By contradiction, suppose that

$$\sup_{\mathcal{I}\times[0,T]\times\mathbb{R}^d} \tilde{u}_\cdot - \tilde{w}_\cdot > 0 \; . \tag{1.4.78}$$

Due to the growth condition on $\tilde{u}_\cdot$ and $\tilde{w}_\cdot$, there exist $R > 0$ such that

$$\tilde{u}_i(t,x) - \tilde{w}_i(t,x) \quad<\quad 0 \; , \tag{1.4.79}$$

for all $(i,t,x) \in \mathcal{I}\times[0,T]\times\mathbb{R}^d$ such that $|x| \geq R$. Then, from (1.4.72) and since $u_\cdot - \tilde{w}_\cdot$ is upper semicontinuous, there exist $(i_0,t_0,x_0) \in \mathcal{I}\times[0,T]\times\mathbb{R}^d$ such that

$$\sup_{(i,t,x)\in\mathcal{I}\times[0,T]\times\mathbb{R}^d} (\tilde{u}_i - \tilde{w}_i)(t,x) \quad=\quad (\tilde{u}_{i_0} - \tilde{w}_{i_0})(t_0,x_0) \;>\; 0 \; . \tag{1.4.80}$$

Step 3. For $n \geq 1$, we define the function

$$\Theta_n(t,x,y,i) := \tilde{u}_i(t,x) - \tilde{w}_i(t,y) - \varphi_n(t,x,y,i) \; ,$$

with

$$\varphi_n(t,x,y,i) = \frac{n}{2}|x-y|^2 + |x-x_0|^4 + |t-t_0|^2 + \mathbb{1}_{i\neq i_0} \; .$$

for all $(t,x,y,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$. By the growth assumption on $\tilde{u}$ and $\tilde{v}$ and (1.4.72), for all n, there exists $(t_n,x_n,y_n,i_n) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$ attaining the maximum of Θ_n on

$[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$. We have

$$\Theta_n(t_n,x_n,y_n,i_n)\geq\Theta_n(t_0,x_0,x_0,i_0)=(\tilde u_{i_0}-\tilde w_{i_0})(t_0,x_0)\ .$$

By (1.4.79) and (1.4.72), up to a subsequence, (t_n,x_n,y_n,i_n) converge to $(\hat t,\hat x,\hat y,\hat\imath)$. Sending n to infinity provides

$$\begin{aligned}\bar\ell:=\limsup_{n\to\infty}\varphi_n(t_n,x_n,y_n,i_n) &\leq \limsup_{n\to\infty}[\tilde u_{i_n}(t_n,x_n)-\tilde w_{i_n}(t_n,y_n)-(\tilde u_{i_0}-\tilde w_{i_0})(t_0,x_0)]\\ &\leq \tilde u_{\hat\imath}(\hat t,\hat x)-\tilde w_{\hat\imath}(\hat t,\hat y)-(\tilde u_{i_0}-\tilde w_{i_0})(t_0,x_0)\ .\end{aligned}$$

In particular, $\bar\ell<+\infty$ and $\hat x=\hat y$. Using the definition of (t_0,x_0,i_0) as a maximizer of $\tilde u_\cdot-\tilde w_\cdot$, we see that:

$$0\ \leq\ \bar\ell\ \leq\ (\tilde u_{\hat\imath}-\tilde w_{\hat\imath})(\hat t,\hat x)-(\tilde u_{i_0}-\tilde w_{i_0})(t_0,x_0)\ \leq\ 0\ ,$$

which implies

$$\begin{aligned}(t_n,x_n,y_n,i_n) &\to (t_0,x_0,x_0,i_0)\ , &(1.4.81)\\ n|x_n-y_n|^2 &\to 0\ , &(1.4.82)\\ \tilde u_{i_n}(t_n,x_n)-\tilde w_{i_n}(t_n,y_n) &\to (\tilde u_{i_0}-\tilde w_{i_0})(t_0,y_0)\ . &(1.4.83)\end{aligned}$$

Being $\mathcal{I}$ endowed with the discrete topology, we can assume $i_n=i_0$ for all $n\geq 1$.

Step 4. We now show that for n large enough

$$\tilde u_{i_0}(t_n,x_n)-\sup_{0\leq k<\bar K}\tilde u_{i_0k}(t_n,x_n)\ >\ 0\ . \qquad (1.4.84)$$

On the contrary, up to a subsequence, we would have for all n,

$$\tilde u_{i_0}(t_n,x_n)-\sup_{0\leq k<\bar K}\tilde u_{i_0k}(t_n,x_n)\ \leq\ 0\ . \qquad (1.4.85)$$

Moreover, by the viscosity supersolution property of $\tilde w$ to (1.4.77), we have

$$\tilde w_{i_0}(t_n,y_n)-\sup_{0\leq k<\bar K}\tilde w_{i_0k}(t_n,y_n)\ \geq\ \Delta_{i_0}\ >\ 0\ .$$

We deduce from the two previous inequalities

$$\begin{aligned}\tilde u_{i_0}(t_n,x_n)-\sup_{0\leq k<\bar K}\tilde u_{i_0k}(t_n,x_n) &\leq \tilde w_{i_0}(t_n,y_n)-\sup_{0\leq k<\bar K}\tilde w_{i_0k}(t_n,y_n)-\Delta_{i_0}\\ \tilde u_{i_0}(t_n,x_n)-\tilde w_{i_0}(t_n,y_n)+\Delta_{i_0} &\leq \sup_{0\leq k<\bar K}\tilde u_{i_0k}(t_n,x_n)-\sup_{0\leq k<\bar K}\tilde w_{i_0k}(t_n,y_n)\\ &\leq \sup_{0\leq k<\bar K}[\tilde u_{i_0k}(t_n,x_n)-\tilde w_{i_0k}(t_n,y_n)]\ . \qquad (1.4.86)\end{aligned}$$

Since $\Delta_{i_0}>0$, for all n there exists k_n such that

$$\sup_{0\leq k<\bar K}[\tilde u_{i_0k}(t_n,x_n)-\tilde w_{i_0k}(t_n,y_n)]-\frac{\Delta_{i_0}}{2}\leq\tilde u_{i_0k_n}(t_n,x_n)-\tilde w_{i_0k_n}(t_n,y_n)\ .$$

From (1.4.72), up to a subsequence, we may assume that $(k_n)_n$ converges to k_0 in $\mathbb{N}$. Hence, by sending n to infinity into (1.4.86), it follows with (1.4.83) and the upper (resp. lower)-semicontinuity of $\tilde{u}$ (resp. $\tilde{w}$) that :

$$(\tilde{u}_{i_0} - \tilde{w}_{i_0})(t_0, x_0) + \frac{\Delta_{i_0}}{2} \leq (\tilde{u}_{i_0 k_0} - \tilde{w}_{i_0 k_0})(t_0, x_0) ,$$

which is a contradiction to (1.4.80).

Step 5. Let us check that, up to a subsequence, $t_n < T$ for all n. On the contrary, $t_n = t_0 = T$ for n large enough, and from (1.4.84), the viscosity subsolution property of $\tilde{u}$ to (1.4.74) and (1.4.67), we would get

$$\tilde{u}_{i_0}(T, x_n) \leq \tilde{g}_{i_0}(x_n) .$$

On the other hand, by the viscosity supersolution property of $\tilde{w}$ to (1.4.74), we have $\tilde{w}(T, y_n) \geq \tilde{g}_{i_0}(y_n)$, and so

$$\tilde{u}_{i_0}(T, x_n) - \tilde{w}_{i_0}(T, y_n) \leq \tilde{g}_{i_0}(x_n) - \tilde{g}_{i_0}(y_n) .$$

By sending n to infinity, and from Assumption (A2) and (1.4.83), this would imply $\tilde{u}_{i_0}(t_0, x_0) - \tilde{w}_{i_0}(t_0, x_0) \leq 0$, a contradiction to (1.4.78).

Step 6. We may, then, apply Ishii's lemma (see, *e.g.*, [48, Theorem 8.3]) to $(t_n, x_n, y_n) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ that attains the maximum of $\Theta_n(., i_0)$ and we get $(p^n_{\tilde{u}}, q^n_{\tilde{u}}, M_n) \in \bar{J}^{2,+}\tilde{u}_{i_0}(t_n, x_n)$ and $(p^n_{\tilde{w}}, q^n_{\tilde{w}}, N_n) \in \bar{J}^{2,-}\tilde{w}_{i_0}(t_n, y_n)$ such that

$$\begin{aligned} p^n_{\tilde{u}} - p^n_{\tilde{w}} &= \partial_t \varphi_n(t_n, x_n, y_n, i_0) = 2(t_n - t_0) , \\ q^n_{\tilde{u}} &= D_x \varphi_n(t_n, x_n, y_n, i_0) = n(x_n - y_n) + 4(x_n - x_0)|x_n - x_0|^2 , \\ q^n_{\tilde{w}} &= -D_y \varphi_n(t_n, x_n, y_n, i_0) = n(x_n - y_n) , \end{aligned}$$

and

$$\begin{pmatrix} M_n & 0 \\ 0 & -N_n \end{pmatrix} \leq A_n + \frac{1}{2n} A_n^2 , \tag{1.4.87}$$

where

$$A_n := D^2_{(x,y)} \varphi_n(t_n, x_n, y_n, i_0) = n \begin{pmatrix} \mathbb{I}_d & -\mathbb{I}_d \\ -\mathbb{I}_d & \mathbb{I}_d \end{pmatrix} - \begin{pmatrix} 4|x_n - x_0|^2 \mathbb{I}_d + 8(x_n - x_0)(x_n - x_0)^\top & \mathbb{O}_d \\ \mathbb{O}_d & \mathbb{O}_d \end{pmatrix} ,$$

with $\mathbb{I}_d$ and $\mathbb{O}_d$ respectively the identity and the zero matrix of $\mathbb{R}^{d \times d}$. A straightforward computation gives

$$A_n + \frac{1}{2n} A_n^2 = 2n \begin{pmatrix} \mathbb{I}_d & -\mathbb{I}_d \\ -\mathbb{I}_d & \mathbb{I}_d \end{pmatrix} - \begin{pmatrix} 2A'_n & -\frac{1}{2} A'_n \\ -\frac{1}{2} A'_n & \mathbb{O}_d \end{pmatrix} + \frac{1}{2n} \begin{pmatrix} (A'_n)^2 & \mathbb{O}_d \\ \mathbb{O}_d & \mathbb{O}_d \end{pmatrix} ,$$

where $A'_n := 4|x_n - x_0|^2 \mathbb{I}_d + 8(x_n - x_0)(x_n - x_0)^\top$. Since the matrix A'_n is positive, we have that the following matrix

$$\begin{pmatrix} 2A'_n & -\frac{1}{2} A'_n \\ -\frac{1}{2} A'_n & 2A'_n \end{pmatrix}$$

is positive as well. Therefore, we can bound the right-hand side of (1.4.87) by

$$A_n + \frac{1}{2n}A_n^2 \quad \leq \quad 2n\begin{pmatrix} \mathbb{I}_d & -\mathbb{I}_d \\ -\mathbb{I}_d & \mathbb{I}_d \end{pmatrix} + \tilde{A}_n \tag{1.4.88}$$

with

$$\tilde{A}_n \quad = \quad \begin{pmatrix} \frac{1}{2n}(A_n')^2 & \mathbb{O}_d \\ \mathbb{O}_d & 2A_n' \end{pmatrix} ,$$

is such that $\limsup_{n\to\infty}\frac{1}{|x_n-x_0|^2}|\tilde{A}_n| < +\infty$. From the viscosity supersolution property of $\tilde{w}_{i_0}$ to (1.4.73), we have

$$\eta\tilde{w}_{i_0}(t_n,y_n) - p^n_{\tilde{w}} + \tilde{F}^*(t_n,y_n,\tilde{w}_{i_0}(t_n,y_n),q^n_{\tilde{w}}) - \lambda(y_n)^\top q^n_{\tilde{w}} - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(y_n)N_n) \geq 0 .$$

On the other hand, from (1.4.84) and the viscosity subsolution property of $\tilde{u}$ to (1.4.73), we have

$$\eta\tilde{u}_{i_0}(t_n,x_n) - p^n_{\tilde{u}} + \tilde{F}_*(t_n,x_n,\tilde{u}_{i_0}(t_n,x_n),q^n_{\tilde{u}}) - \lambda(x_n)^\top q^n_{\tilde{u}} - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x_n)M_n) \leq 0 .$$

By subtracting the two previous inequalities, we obtain

$$\begin{aligned}\eta(\tilde{u}_{i_0}(t_n,x_n) - \tilde{w}_{i_0}(t_n,y_n)) \quad &\leq \quad p^n_{\tilde{u}} - p^n_{\tilde{w}} + \tilde{F}^*(t_n,y_n,\tilde{w}_{i_0}(t_n,y_n),q^n_{\tilde{w}}) - \tilde{F}_*(t_n,x_n,\tilde{u}_{i_0}(t_n,x_n),q^n_{\tilde{u}}) + \\ &\quad +\lambda(x_n)^\top q^n_{\tilde{u}} - \lambda(y_n)^\top q^n_{\tilde{w}} + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x_n)M_n) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(y_n)N_n) \\ &= \quad p^n_{\tilde{u}} - p^n_{\tilde{u}} + \Delta C^1_n + \Delta C^2_n + \Delta C^3_n ,\end{aligned} \tag{1.4.89}$$

where

$$\begin{aligned}\Delta C^1_n \quad &:= \quad \tilde{F}^*(t_n,y_n,\tilde{w}_{i_0}(t_n,y_n),q^n_{\tilde{w}}) - \tilde{F}_*(t_n,x_n,\tilde{u}_{i_0}(t_n,x_n),q^n_{\tilde{u}}) ,\\ \Delta C^2_n \quad &:= \quad \lambda(x_n)^\top q^n_{\tilde{u}} - \lambda(y_n)^\top q^n_{\tilde{w}} ,\\ \Delta C^3_n \quad &:= \quad \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x_n)M_n) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(y_n)N_n) .\end{aligned}$$

From (1.4.81)), we have $p^n_{\tilde{u}} - p^n_{\tilde{u}} \to 0$ as $n \to 0$. From the Lipschitz continuity of λ and (1.4.82), we have $\Delta C^2_n \to 0$ as $n \to 0$. From (1.4.81), (1.4.82), (1.4.88) and the Lipschitz property of σ, we also have $\Delta C^3_n \to 0$ as $n \to 0$.

Fix $\varepsilon,\eta > 0$, then, the exists (x',r',p') and (x'',r'',p'') such that

$$\sup_{\substack{|y_n - x| \leq \varepsilon \\ |\tilde{w}_{i_0}(t_n,y_n) - r| \leq \varepsilon \\ |q^n_{\tilde{w}} - p| \leq \varepsilon}} \sup_{a\in\mathcal{N}_\varepsilon(x,p)}\left\{\tilde{\lambda}_Y(x,r,a)\right\} - \inf_{\substack{|x_n - x| \leq \varepsilon \\ |\tilde{u}_{i_0}(t_n,x_n) - r| \leq \varepsilon \\ |q^n_{\tilde{u}} - p| \leq \varepsilon}} \sup_{a\in\mathcal{N}_\varepsilon(x,p)}\left\{\tilde{\lambda}_Y(x,r,a)\right\} \quad \leq$$

$$2\eta + \sup_{a\in\mathcal{N}_\varepsilon(x',p')}\left\{\tilde{\lambda}_Y(x',r',a)\right\} - \sup_{a\in\mathcal{N}_\varepsilon(x'',p'')}\left\{\tilde{\lambda}_Y(x'',r'',a)\right\} \qquad ,$$

with

$$\begin{matrix} |y_n - x'| \leq \varepsilon \\ |(\tilde{w}_{i_0}(t_n,y_n) - r'| \leq \varepsilon \\ |q^n_{\tilde{w}} - p'| \leq \varepsilon \end{matrix} \qquad \text{and} \qquad \begin{matrix} |x_n - x''| \leq \varepsilon \\ |\tilde{u}_{i_0}(t_n,x_n) - r''| \leq \varepsilon \\ |q^n_{\tilde{u}} - p''| \leq \varepsilon \end{matrix}$$

Since $\tilde{\lambda}_Y$ is nondecreasing in its second argument, by following the same argument as in the proof of Lemma 1.4.2, we get from (1.4.80), Assumptions A1 and (A4)*(iii)* and (1.4.81), that

$$\limsup_{n\to+\infty} \Delta C_n^1 \quad \leq \quad 0 \,.$$

Therefore, by sending $n \to \infty$ into (1.4.89), we conclude with (1.4.83) that $\eta(\tilde{u}_{i_0}(t_0, x_0) - \tilde{w}_{i_0}(t_0, y_0)) \leq 0$, a contradiction with (1.4.80). □

From Theorems 1.4.4, 1.4.5 and 1.4.6, we get the following characterisation of the function $\bar{v}$.

Corollary 1.4.1. *Suppose that $\bar{v}$ satisfies*

$$\sup_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}} \frac{|\bar{v}_i(t,x)|}{1+|x|^\gamma} < +\infty \quad , \tag{1.4.90}$$

for some $\gamma > 0$ and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\bar{v}_i(t,x)| \xrightarrow[|i|\to\infty]{} 0 \,. \tag{1.4.91}$$

Under Assumptions A1-A2-A3-A4, $\bar{v}$ is the unique viscosity solution to (1.4.33)-(1.4.66) *satisfying* (1.4.90)-(1.4.91)*. Moreover, $\bar{v}$ is continuous on* $[0, T) \times \mathbb{R}^d \times \mathcal{I}$*.*

We recall that Section 1.2.3 provides an example of a value function satisfying conditions (1.4.33)-(1.4.66).

1.5 Appendix

Proposition 1.5.9. *For $\ell \geq 1$, E_ℓ is a closed subset of $\mathcal{M}_F(\mathcal{I}\times\mathbb{R}^\ell)$ for the topology of the weak convergence of measures.*

Proof. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of E_ℓ such that $\mu_n = \sum_{i\in V_n} \delta_{(i,x_n^i)} \xrightarrow{w} \mu \in \mathcal{M}_F(\mathcal{I}\times\mathbb{R}^\ell)$. We prove that μ is an element of E_ℓ, *i.e.*, it can be written as $\mu = \sum_{i\in V} \delta_{(i,x^i)}$ for some set $V \subseteq \mathcal{I}$, $|V| < \infty$ and some points $(x_i)_{i\in V}$.

Consider the continuous functions $\mathbb{1}_{\{i\}\times\mathbb{R}^\ell}$, for $i \in \mathcal{I}$. We, then, have

$$\langle \mu_n \ , \ \mathbb{1}_{\{i\}\times\mathbb{R}^\ell} \rangle = \begin{cases} 1 & \text{if } i \in V_n \\ 0 & \text{if } i \notin V_n \end{cases} \,.$$

For each $i \in \mathcal{I}$, we have that the sequence $\big(\langle \mu_n \ , \ \mathbb{1}_{\{i\}\times\mathbb{R}^\ell}\rangle\big)_n$ is a convergent sequence in $\{0, 1\}$, which is in particular stationary. Let V be defined as follows

$$V := \Big\{ i \in \mathcal{I} \ : \ \langle \mu_n \ , \ \mathbb{1}_{\{i\}\times\mathbb{R}^\ell} \rangle \xrightarrow[n\to\infty]{} 1 \Big\} \,.$$

Let $i \in V$. Since the functions previously described converge, they are constant from a certain rank and there exists $n_i \in \mathbb{N}$ such that for $n \geq n_i$ we have $i \in V_n$. For $f \in C_b(\mathbb{R}^\ell)$ and consider the function $\mathbb{1}_{\{i\}} \otimes f : \mathcal{I}\times\mathbb{R}^\ell \to \mathbb{R}$. We have

$$f(x_n^i) = \langle \mu_n, \mathbb{1}_{\{i\}} \otimes f \rangle \longrightarrow \langle \mu, \mathbb{1}_{\{i\}} \otimes f \rangle \in \mathbb{R} \,.$$

This means that, for each $i \in V$ and $f \in C_b(\mathbb{R}^\ell)$, the sequence $(f(x_n^i))_n$ converges, thus $(x_n^i)_n$ converges to a point $x^i \in \mathbb{R}^\ell$. Indeed, if $|x_n^i| \to +\infty$ as $n \to \infty$, we would have $\langle \mu_n, \mathbb{1}_{\{i\}} \otimes f_p \rangle = 0$, for n large enough, for any $p \geq 1$, where $f_p \in C_b(\mathbb{R}^\ell)$ is such that $f_p \geq 0$, $f_p(x) = 1$ for $|x| \leq p$ and $f_p(x) = 0$ for $|x| \geq p+1$. This gives $\langle \mu, \mathbb{1}_{\{i\}} \otimes f_p \rangle = 0$, for any p, and, by the monotone convergence theorem we get $\langle \mu, \mathbb{1}_{\{i\}\times\mathbb{R}^\ell} \rangle = 0$ which contradict the weak convergence of μ_n to μ. Therefore, the sequence $(x_n^i)_n$ is bounded and the convergence of $(f(x_n^i))_n$ for $f \in C_b(\mathbb{R}^\ell)$ implies the convergence of $(x_n^i)_n$.

We, then, notice that, for any continuous and bounded function f on $\mathcal{I} \times \mathbb{R}^\ell$, the functions $f_i := f(i, \cdot)$ are continuous and bounded on $\mathbb{R}^\ell$, for $i \in \mathcal{I}$. This entails that

$$\int_{\mathcal{I}\times\mathbb{R}^\ell} f d\mu_n \quad = \quad \sum_{i\in V} f_i(x_n^i) \, ,$$

for n large enough, and

$$\int_{\mathcal{I}\times\mathbb{R}^\ell} f d\mu_n \quad \xrightarrow[n\to+\infty]{} \quad \int_{\mathcal{I}\times\mathbb{R}^\ell} f d\left(\sum_{i\in V} \delta_{(i,x^i)}\right) \, .$$

This means that we have $\mu = \sum_{i\in V} \delta_{(i,x^i)}$.

Finally, to prove that $\mu \in E_\ell$, we show that there do not exist $i, j \in V$ such that $i \prec j$. Fix $i, j \in V$. From the previous steps, there exists some n such that $i, j \in V_n$. Since $\mu_n \in E_\ell$, we get $i \not\prec j$ and $j \not\prec i$. Therefore, $\mu \in E_\ell$. □

Chapter 2

Optimal Stopping of Branching Diffusion Processes

Outline of the current chapter

This chapter is a joint work with Idris Kharroubi.

Abstract: This article explores an optimal stopping problem for branching diffusion processes. It consists in looking for optimal stopping lines, a type of stopping time that maintains the branching structure of the processes under analysis. By using a dynamic programming approach, we characterize the value function for a multiplicative cost that depends on the particle's label. We reduce the problem's dimensionality by setting a branching property and defining the problem in a finite-dimensional context. Within this framework, we focus on the value function, establishing polynomial growth and local Lipschitz properties, together with an innovative dynamic programming principle. This outcome leads to an analytical characterization with the help of a nonlinear elliptic PDE. We conclude by showing that the value function serves as the unique viscosity solution for this PDE, generalizing the comparison principle to this setting.

2.1 Introduction

The class of the branching diffusion processes is an object that received a great deal of interest since its introduction in the late sixties in [94, 95, 96, 144]. This class is used to describe the evolution of a population where we are interested in a special feature, e.g. the spatial motion, of identical particles that reproduce at random times.

In the study of branching diffusion processes, a fundamental question emerges: at what juncture does it become optimal to halt such a process? This question delves into the determination of an opportune point in time to stop the evolution of a branching diffusion. This research line echoes the optimization of a given functional to trade-off between the diffusion and reproduction of these processes and a possible degradation of the reward. By investigating the optimal stopping time for branching diffusion processes, we aim to shed light on the decision-making process involved in terminating these dynamic systems, thereby enhancing our understanding of their behavior and enabling more effective applications in various fields of study.

One possible approach to consider is looking at the entire branching diffusion process as a whole, as done in Chapter 3, and finding a universal stopping time that applies to all active branches simultaneously. This global stopping time serves as a comprehensive decision rule, enabling a synchronized halt to the progression of each branch in the system, regardless of their characteristics or temporal disparities.

Although the aforementioned approach has its appeal, it may not fully align with the intrinsic structure of such processes. Indeed, the fundamental nature of a branching process, even when studied as a collective entity, is fundamentally rooted in its ability to portray the trajectory and dynamics of a singular individual. Therefore, while a global perspective may offer valuable insights and provide a comprehensive overview of the system, it may inadvertently disregard the inherent individuality of the branches.

This dual mode between the individuality of the single component as opposed to the wholeness of the population is a key concept in cooperative game theory. For example, mean-field control literature (see, *e.g.*, [31, 32]) deals with the control of large-scale systems involving a multitude of interacting agents, assumed to be rational decision-makers who aim to optimize their objective functions. The goal is to find control strategies that maximize a specific objective at the population level, which aligns with the optimal behavior of each agent, influenced by the collective behavior of the entire population. An additional example illustrating the transformation of global behavior into individual optimization can be observed in [42] and Chapters 1 and 4. These studies prove how control strategies are contingent upon the decisions made by each participant. Moreover, the concept of the branching property emerges as a means to reduce the complexity of the problem, consequently shifting the focus toward analyzing the dynamics of the individual agents.

To capture the decision-making process of individuals within a collective framework, we adopt the concept of stopping lines. This mathematical object, introduced in [37, 38], serves as the counterpart to stopping times in branching dynamics. Stopping lines are characterized by a subset of the process's genealogy, where no member can be traced back to another member, and we can see their use in applications such as [110].

Although stopping lines have been used in previous studies, the exploration of optimal stopping lines based on specific criteria remains, to the best of our knowledge, an open problem. This article aims to address this research gap by directing our attention to this exact issue.

Within a branching diffusion process framework, we look for the characterization of the value function linked to an infinite horizon optimal stopping problem. Optimization is done over the set of stopping lines, where each branch becomes eligible for halting only if no preceding ancestor has been stopped before. We narrow our investigation to multiplicative rewards, similar to the approach taken in [42, 125]. Drawing inspiration from [39], we prove a fundamental branching property. This property provides conditional independence among the offspring branches subsequent to a given conditioning time. This allows working within a finite-dimensional setting, distinguishing it from the traditional approach that treats branching diffusion dynamics as measure-valued processes. This framework, additionally, yields polynomial growth and local Lipschitz properties for the value function.

We employ a dynamic programming approach to characterize the value function as a solution to a specific Partial Differential Equation (PDE). Establishing an original Dynamic Programming Principle (DPP), we extend the framework of the classical optimal stopping problem to our branching context. This outcome paves the way for an analytical characterization of the value function.

The corresponding PDE takes the form of an obstacle problem with a semilinear term, which involves a polynomial series associated with the branching mechanism and value functions related to offspring labels. Assuming that this series has an infinite radius of convergence, we show that the value function is a solution in the sense of viscosity to this PDE. It is worth noting that a global bound on the label for the test functions is needed within the viscosity properties. This condition serves to retrieve the martingale property for the compensated jump component of the branching diffusion dynamics.

To conclude the PDE characterization, we present a comparison theorem. The presence of the semilinear term, tied to the value functions associated with offspring labels, introduces a non-classical aspect to this PDE. We explore a multiplicative penalization, making the viscosity solutions go towards zero in the spatial variable as a result of the previously demonstrated polynomial growth. Then, using the assumption of vanishing rewards as the label goes to infinity, we establish the comparison principle for value functions related to sufficiently large starting labels. We finally extend this analysis to cover the remaining functions through a backward induction on the size of the label.

The remainder of the paper is structured as follows. Section 2.2 presents a detailed description of the model under examination, focusing on the characteristics of branching diffusion processes and stopping lines. Additionally, we discuss the continuity of these processes' trajectories and highlight a crucial branching property that will play a significant role in subsequent sections. In Section 2.3, we introduce the optimal stopping problem and establish the regularity of the corresponding value function. Section 2.4 is dedicated to proving the dynamic programming principle, while Section 2.5 provides the characterization of the value function as the unique viscosity solution to an obstacle problem.

2.2 Branching diffusion processes formulation

Label set We start by introducing the Ulam–Harris–Neveu notation. This is key in the description of the tree structure of the problem, identifying immediately the genealogy of a particle. For $n \geq 1$, we write $i = i_1 \dots i_n$ for the multi-integer $i = (i_1, \dots, i_n) \in \mathbb{N}^n$. For $n, m \geq 1$ and two multi-integers $i = i_1 \dots i_n \in \mathbb{N}^n$ and $j = j_1 \dots j_m \in \mathbb{N}^m$, we define their concatenation $ij \in \mathbb{N}^{n+m}$ as

$$ij \quad := \quad i_1 \dots i_n j_1 \dots j_m \ . \tag{2.2.1}$$

The evolution of the particle population can now be described with the help of the set of labels $\mathcal{I}$ defined as follows

$$\mathcal{I} \quad = \quad \{\varnothing\} \cup \bigcup_{n=1}^{+\infty} \mathbb{N}^n \ ,$$

where the label $\varnothing$ corresponds to the mother particle. We extend the concatenation (2.2.1) to the whole set $\mathcal{I}$ with $\varnothing i = i\varnothing = i$, for all $i \in \mathcal{I}$.

When the particle $i = i_1 \dots i_n \in \mathbb{N}^n$ gives birth to k particles, the off-springs are labelled $i0, \dots, i(k-1)$. By employing this method of generating the genealogy, we can establish a partial ordering $\preceq$ (resp. $\prec$) by

$$i \preceq j \quad \Leftrightarrow \quad \exists \ell \in \mathcal{I} \ : \ i = j\ell \qquad (\text{resp. } i \prec j \Leftrightarrow \exists \ell \in \mathcal{I} \setminus \{\varnothing\} \ : \ i = j\ell)$$

for all $i, j \in \mathcal{I}$. We say that $i \in \mathcal{I}$ is the parent of $j \in \mathcal{I}$ if $j = i\ell$ with $\ell \in \mathbb{N}$. Moreover, if $i = i_1 \dots i_n$, we say that i belongs to the n-th generation of the population.

We endow $\mathcal{I}$ with the discrete topology, which is generated by the following distance $d_{\mathcal{I}}$

$$d^{\mathcal{I}}(i,j) = \sum_{\ell=p+1}^{n} (i_\ell + 1) + \sum_{\ell'=p+1}^{m} (j_{\ell'} + 1) \ ,$$

for $i = i_1 \cdots i_n \in \mathbb{N}^n, j = j_1 \cdots j_m \in \mathbb{N}^m$, where p is the generation of the greatest common ancestor, i.e., $p = \max\{\ell \geq 1 : i_\ell = j_\ell\}$. We next write $|i| := d^{\mathcal{I}}(i, \varnothing)$ for $i \in \mathcal{I}$. Define the following function $\mathfrak{g} : \mathcal{I} \to \mathbb{N}$ such that $\mathfrak{g}(i) = n$ for $i = i_1 \cdots i_n \in \mathbb{N}^n$, which corresponds to the generation of the particle i.

Set of marked trees We will say tree to describe the family tree of the population. A tree ω^0 is a subset of $\mathcal{I}$ that satisfies the following properties: $\varnothing \in \omega^0$,

$$ij \in \omega^0 \quad \Rightarrow \quad i \in \omega^0 \quad \text{ for } \ i, j \in \mathcal{I} \ ,$$

and for any $i \in \omega^0$, there exists $\nu_i(\omega^0) \in \mathbb{N}$ such that

$$i\ell \in \omega^0 \quad \Rightarrow \quad 0 \leq \ell \leq \nu_i(\omega^0) - 1 \tag{2.2.2}$$

with the convention that $\nu_i(\omega^0) = 0$, when $i\ell \notin \omega^0$ for all $\ell \in \mathbb{N}$.

We denote Ω^0 the set of trees. We say that $i \in \mathcal{I}$ is a node of $\omega^0 \in \Omega^0$ if $i \in \omega^0$. For $i \in \mathcal{I}$, let Ω_i^0 be the subset of trees having i as a node, i.e.,

$$\Omega_i^0 \quad := \quad \left\{\omega^0 \in \Omega^0 \ : \ i \in \omega^0\right\} .$$

We notice that Ω_i^0 is the domain of the map ν_i introduced in (2.2.2) for $i \in \mathcal{I}$.

For $d \in \mathbb{N}^*$, let $\Omega^1 := C^0(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}_+$ be the space of marks. We denote B (resp. ρ) the projection map from Ω^1 to its first (resp. second) component, that is

$$B(\omega^1, s) = \omega^{10}(s), \qquad \rho(\omega^1) = \omega^{11}$$

for $\omega^1 = (\omega^{10}, \omega^{11}) \in \Omega^1$ with $\omega^{10} \in \Omega^{10}$ and $\omega^{11} \in \Omega^{11}$.

The set of marked trees Ω is now defined as

$$\Omega \quad := \quad \Big\{\omega = \big(\omega^0, (\omega_i^1 \ , \ i \in \omega^0)\big) \ , \ \omega^0 \in \Omega^0 \ , \ \omega_i^1 \in \Omega^1\Big\},$$

and we denote π^0 the canonical projection from Ω to Ω^0. For $i \in \mathcal{I}$, we set $\Omega_i = (\pi^0)^{-1}(\Omega_i^0)$ and still denote ν_i the map induced on Ω_i by π^0 and (2.2.2). For $i \in \mathcal{I}$, we define the canonical projection π_i^1 from Ω_i to Ω^1 by

$$\pi_i^1(\omega) \quad = \quad \omega_i^1 \ , \qquad \text{for } \omega = \big(\omega^0, (\omega_j^1 \ , \ j \in \omega^0)\big) \in \Omega_i \ .$$

We can now extend B (resp. ρ) to Ω_i, obtaining the map B_i (resp. ρ_i) as follows

$$B_i \quad = \quad B \circ \pi_i^1 \ , \qquad \rho_i \quad = \quad \rho \circ \pi_i^1 \ .$$

for $i \in \mathcal{I}$.

One key property of trees is their self-similarity. This means that, when we zoom on a node and look at its offspring, we still have a tree, up to re-indexation of the labels. Therefore, we define the shift operator $T_{i,s}$, for $i \in \mathcal{I}$ and $s \in \mathbb{R}_+$, from Ω_i to Ω. This operator is such that $T_{i,s}(\omega)$ is the subtree of ω starting from a particle i alive at time s. More precisely, we have

$$\begin{aligned}
\pi^0(T_{i,s}(\omega)) &= \{j \in \mathcal{I} \ : \ ij \in \pi^0(\omega)\} \ , \\
\rho_\varnothing(T_{i,s}(\omega)) &= \rho_i(\omega) - s \wedge \rho_i(\omega) \ , \\
B_\varnothing\big(T_{i,s}(\omega), t\big) &= B_i\big(\omega, (s \wedge \rho_i(\omega)) + t\big) - B_i\big(\omega, s \wedge \rho_i(\omega)\big) \ , \ \text{for } t \in [s \wedge \rho_i(\omega), \rho_i(\omega)] \ , \\
B_j(T_{i,s}(\omega)) &= B_{ij}(\omega) \ , \qquad \text{for } \omega \in \Omega_{ij} \ , \ j \neq \varnothing \ , \\
\rho_j(T_{i,s}(\omega)) &= \rho_{ij}(\omega) \ , \qquad \text{for } \omega \in \Omega_{ij} \ , \ j \neq \varnothing \ .
\end{aligned}$$

Lifetime, birthtime, and positions. When dealing with processes indexed on a tree, we have two notions of time to take into consideration. On one hand, the age of the person, and consequently its time of death/reproduction. On the other hand, the calendar time expresses a notion of time for all the particles. Let S_j be the birthtime of a particle $j \in \mathcal{I}$ such that $S_\varnothing = 0$ and, inductively on the generations,

$$S_j = S_i + \rho_i$$

with i the parent of j. With this notion, which encodes the calendar time for the population, we write $\mathcal{V}_t$ for the set of alive particles at time $t \in \mathbb{R}_+$, defined by

$$\mathcal{V}_t = \Big\{ i \in \mathcal{I} \ : \ S_i \leq t < S_i + \rho_i \Big\} \ . \tag{2.2.3}$$

σ-algebrae and filtrations. As σ-algebrae describe the information we can access, we need to define filtration that will match the tree structure. First, on Ω^1, let $\mathcal{H}^1 = (\mathcal{H}^1(t))_{t \in \mathbb{R}_+}$ be the right-continuous filtration generated by marginal projection B and progressively enlarged by ρ:

$$\mathcal{H}_t := \bigcap_{\varepsilon > 0} \sigma\big(B_s, \mathbb{1}_{\rho \leq s} \ , \ s \leq t + \varepsilon\big) \ , \quad t \geq 0 \ .$$

Then, on Ω_i, take $\mathcal{H}_i = (\mathcal{H}_i(t))_{t \in \mathbb{R}_+}$ to be the filtration associated with the evolution of the branch with label i, i.e.,

$$\mathcal{H}_i(t) := (\pi_i^1)^{-1}\big(\mathcal{H}^1(t)\big) \qquad \text{for } t \in \mathbb{R}_+.$$

As done for the birthtime, we consider the σ-algebrae associated with the ancestors of a particle i. Let $\mathcal{G}_\varnothing$ be the completed trivial σ-algebra on Ω. For $j \in \mathcal{I}$, we consider the σ-algebra $\mathcal{G}_j$ on Ω_j defined inductively by

$$\mathcal{G}_j := \sigma(\mathcal{G}_i, \mathcal{H}_i(\rho_i)) \cap \Omega_j$$

with i the parent of j. Finally, we introduce the filtration that stores all the information of the ancestors up to calendar time $t \in \mathbb{R}_+$. Let $\mathcal{A}_i = (\mathcal{A}_i(t))_{t \in \mathbb{R}_+}$ on Ω_i be

$$\mathcal{A}_i(t) := \sigma(\mathcal{G}_i, \mathcal{H}_i(t)), \qquad \text{for } t \in \mathbb{R}_+, i \in \mathcal{I}.$$

We observe that B_i and Z_i are $\mathcal{A}_i$-adapted for $i \in \mathcal{I}$. We finally endow Ω with the σ-algebra $\mathcal{F}$

generated by $\mathcal{G}_i$ for $i \in \mathcal{I}$.

Finally, we consider the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ generated by all the particles alive, with respect to the calendar time $t \in \mathbb{R}_+$

$$\mathcal{F}_t := \sigma\left(\mathcal{A}_i(t - S_i) : i \in \mathcal{V}_t\right).$$

Stopping lines A stopping line is a collection of maps $(\tau_i,\ i \in \mathcal{I})$ such that

(i) $\tau_i:\ \Omega_i \to \mathbb{R}_+$ is a $\mathcal{A}_i$-stopping time for all $i \in \mathcal{I}$,

(ii) the random set L_τ defined by

$$L_\tau(\omega) \quad = \quad \Big\{i \in \pi^0(\omega)\ :\ 0 \le \tau_i(\omega) < \rho_i(\omega)\Big\}\ , \quad \omega \in \Omega\ ,$$

satisfies the so-called *line property*

$$i, j \in \mathcal{I}\ ,\ j \prec i \ \text{ and } \ i \in L_\tau \quad \Rightarrow \quad j \notin L_\tau\ .$$

This last property tells that the set L_τ cannot select two particles if one is the ancestor of the other.

We denote the set of stopping lines $\mathcal{SL}$. For $\tau \in \mathcal{SL}$, we define the set D_τ as

$$D_\tau \quad := \quad \{i \in \mathcal{I}\ :\ \exists j \in \mathcal{I}\ ,\ j \prec i\ ,\ j \in L_\tau\}\ ,\ .$$

which corresponds to the set of strict descendants of the line L_τ. On $\mathcal{SL}$, we consider the following partial order

$$\tau \le \theta \quad \Leftrightarrow \quad D_\theta \subset D_\tau \text{ and } [i \in L_\tau \cap L_\theta \Rightarrow \tau_i \le \theta_i]$$

for $\tau, \tau' \in \mathcal{SL}$. As for stopping times on the real line, the σ-algebra $\mathcal{F}_\tau$ related to a stopping line τ is defined as

$$\mathcal{F}_\tau \quad := \quad \sigma\Big(\{i \notin D_\tau\} \cap \mathcal{A}_i(\tau_i)\ ,\ i \in \mathcal{I}\Big)\ .$$

With respect to the filtration $\mathbb{F}$, we see that $\mathcal{F}_t$ corresponds to the filtration generated by the stopping line τ^t

$$\begin{aligned} \tau^t &:= t - S_i, && \text{if } i \in \mathcal{V}_t, \\ \tau^t &:= \rho_i, && \text{else.} \end{aligned}$$

Moreover, we have $L_{\tau^t} = \mathcal{V}_t$.

Probability law and branching property We turn to the definition of the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We follow the construction of [124] for Galton Watson Processes and extend to Brownian branching processes by [39]. For that, we introduce the auxiliary space

$$\Omega^* \quad = \quad \Big(C^0(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}_+ \times \mathbb{N}\Big)^{\mathcal{I}}$$

that we endow with the σ-algebra

$$\mathcal{F}^* = \Big(\mathcal{B}(C^0(\mathbb{R}_+,\mathbb{R}^m))\otimes\mathcal{B}(\mathbb{R}_+))^{\otimes\mathbb{N}}\Big)^{\otimes\mathcal{I}}$$

We then define on $(\Omega^*,\mathcal{F}^*)$ the probability measure $\mathbb{P}^*$ by

$$\mathbb{P}^* = \Big(\mathbb{P}^0\otimes\mathcal{E}(\alpha)\otimes\sum_{n\in\mathbb{N}}p_n\delta_n\Big)^{\otimes\mathcal{I}},$$

where $\mathbb{P}^0$ stands for the Wiener measure on $C^0(\mathbb{R}_+,\mathbb{R}^m)$. For $i\in\mathcal{I}$, we define on Ω^* the projections ν_i^*, B_i^* and ρ_i^* by

$$\begin{aligned} B_i^*(\omega^*) &= \omega_i^{*,1} \in C^0(\mathbb{R}_+,\mathbb{R}^m)\,,\\ \rho_i^*(\omega^*) &= \omega_i^{*,2} \in \mathbb{R}_+\,,\\ \nu_i^*(\omega^*) &= \omega_i^{*,3} \in \mathbb{N}\,,\end{aligned}$$

for $\omega^*=\big(\omega_i^{*,1},\omega_i^{*,2},\omega_i^{*,3}\big)_{i\in\mathcal{I}}\in\Omega^*$. We next define the map $\Phi:\ \Omega^*\to\Omega$ such that for $\omega^*\in\Omega^*$, $\Phi(\omega^*)$ is the tree of Ω starting from $B_\varnothing(\omega^*)$ such that at each node $i\in\mathcal{I}$, it has $\nu_i^*(\omega^*)$ offspring with $B_{i0}^*,\ldots,B^*_{i(\nu_i^*(\omega^*)-1)}$ trajectories and $\rho_{i0}^*,\ldots,\rho^*_{i(\nu_i^*(\omega^*)-1)}$ respective extinction times. We put on $(\Omega,\mathcal{F})$ the probability measure $\mathbb{P}$ defined as the image measure of $\mathbb{P}^*$ by Φ.

We have the following result on the laws of B_i and ρ_i for $i\in\mathcal{I}$.

Proposition 2.2.10. *Given Ω_i, ν_i, B_j and ρ_j, $j\preceq i$ are independent and follow the laws $\sum_n p_n\delta_{\{n\}}$, $\mathbb{P}^0$ and $\mathcal{E}(\alpha)$ for any $i\in\mathcal{I}$.*

Proof. Fix $i=i_1\ldots i_n\in\mathcal{I}$ with $i_1,\ldots,i_n\in\mathbb{N}$, $A_j\times B_j\in\mathcal{B}(C^0(\mathbb{R}_+,\mathbb{R}^m))\times\mathcal{B}(\mathbb{R}_+)$ for $j\preceq i$ and $k\in\mathbb{N}$. We then have

$$\begin{aligned} \mathbb{P}\big((B_j,\rho_j)\in A_j\times B_j,\ j\preceq i\,,\nu_i=k\mid\Omega_i\big) &=\\ \frac{\mathbb{P}\big((B_j,\rho_j)\in A_j\times B_j\ ,j\preceq i,\ \nu_i=k,\ \ \nu_\varnothing\geq i_1+1,\ldots,\ \nu_{i_1\ldots i_{n-1}}\geq i_n+1\big)}{\mathbb{P}\big(\nu_\varnothing\geq i_1+1,\ldots,\ \nu_{i_1\ldots i_{n-1}}\geq i_n+1\big)} &=\\ \frac{\mathbb{P}^*\big((B_j^*,\rho_j^*)\in A_j\times B_j\ ,j\preceq i,\ \nu_i^*=k,\ \nu_\varnothing^*\geq i_1+1,\ldots,\ \nu^*_{i_1\ldots i_{n-1}}\geq i_n+1\big)}{\mathbb{P}\big(\nu^*_\varnothing\geq i_1+1,\ldots,\ \nu^*_{i_1\ldots i_{n-1}}\geq i_n+1\big)} &=\\ p_k\prod_{j\preceq i}\mathbb{P}^0(A_j)\int_{B_j}\alpha e^{-\alpha u}du \end{aligned}$$

where the last equality comes from the definition of $\mathbb{P}^*$. □

Branching diffusion processes We now define diffusion processes on trees. Let $b:\ \mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\ \mathbb{R}^d\to\mathbb{R}^{d\times m}$ be measurable functions and $x\in\mathbb{R}^m$ a starting point. The branching diffusion process starting from x is a map $X^x:\ \Omega\to\Omega$ such that

$$\pi^0\big(X^x(\omega)\big) = \pi^0(\omega) = \omega^0$$

and

$$\begin{aligned}\rho_i\circ X^x(\omega) &= \rho_i(\omega)\\ B_i\circ X^x(\omega) &= X_i^x(\omega)\end{aligned}$$

for $i \in \mathcal{I}$ and $\omega \in \Omega_i$ where $X_\varnothing$ is defined on Ω by

$$X^x_\varnothing(0) = x \tag{2.2.4}$$
$$dX^x_\varnothing(s) = b(X^x_\varnothing(s))ds + \sigma(X^x_\varnothing(s))dB_\varnothing(s)\ , \quad s \geq 0 \tag{2.2.5}$$

and X^x_i, $i \neq \varnothing$, is defined on Ω_i by

$$X^x_i(0) = X^x_j(\rho_j) \tag{2.2.6}$$
$$dX^x_i(s) = b(X^x_i(s))ds + \sigma(X^x_i(s))dB_i(s)\ , \quad s \geq 0 \tag{2.2.7}$$

for $i \in \mathcal{I}$ where j is the parent of i.

We make the following assumptions on the coefficients b and σ and on the law p.

Assumption A5. *(i) The functions b and σ are Lipschitz continuous, i.e., there exists a constant $L > 0$ such that*

$$|b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leq L|x - x'|, \tag{2.2.8}$$

for all $x, x' \in \mathbb{R}^d$.

(ii) The coefficients p_k, $k \geq 0$, satisfy

$$\sum_{k\geq 0} kp_k = M < +\infty\ .$$

Proposition 2.2.11. *Suppose that Assumption A5 holds.*

(i) There exists a unique process $(X^x_i)_{i\in\mathcal{I}}$ solution to (2.2.4)-(2.2.5)-(2.2.6)-(2.2.7).

(ii) For $p \geq 1$, there exists two constants $\alpha_p > 0$ and $C_p > 0$ such that

$$\mathbb{E}\left[\sup_{s\in[0,\rho_i]} |X^x_i(s)|^{2p}\Big|\Omega_i\right] \leq \left(\sum_{k=1}^{\mathfrak{g}(i)+1} C^k_p\right)\left(1 + |x|^{2p}\right)\ , \tag{2.2.9}$$
$$\mathbb{E}\left[\sup_{s\in[0,\rho_i]} |X^x_i(s) - X^{x'}_i(s)|^{2p}\Big|\Omega_i\right] \leq (C_p)^{\mathfrak{g}(i)+1}|x - x'|^{2p}\ , \tag{2.2.10}$$

for $x, x' \in \mathbb{R}^d$ and $i \in \mathcal{I}$ whenever $\alpha \geq \alpha_p$.

Proof. (i) Since B_i follows $\mathbb{P}^0$ given Ω_i, Assumption A5(i) gives the existence and uniqueness of a process X_i defined on Ω_i satisfying (2.2.6)-(2.2.7) for all $i \in \mathcal{I}$ (see, *e.g.*, [105, Theorem 2.5.7]). This ensures the good definition of the map X^x.
(ii) We turn to the branching property. For that, we prove (2.2.9)-(2.2.10) by induction on the generation.

Fix $x, x' \in \mathbb{R}^d$. From [105, Corollary 10, Section 5, Chapter 2] and [105, Theorem 9, Section 5, Chapter 2], we have that there exists a constant $\bar{C}_p > 0$ that depends only on q and L from the growth condition consequence of (2.2.8) such that

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|X^x_\varnothing(s)\right|^{2p}\right] \leq \bar{C}_p t^{p-1} e^{\bar{C}_p t}\left(1 + |x|^{2p}\right),$$
$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|X^x_\varnothing(s) - X^{x'}_\varnothing(s)\right|^{2p}\right] \leq \bar{C}_p e^{\bar{C}_p t}\left(|x - x'|^{2p}\right)$$

for $t \geq 0$. Since $B_\varnothing$ and $\rho_\varnothing$ are independent and $\rho_\varnothing$ is distributed as an exponential random variable with parameter α, we have that

$$\mathbb{E}\left[\sup_{s\in[0,\rho_\varnothing]} |X^x_\varnothing(s)|^{2p}\right] \leq \bar{C}_p \int_0^\infty t^{p-1} e^{\bar{C}_p t} \alpha e^{-\alpha t} dt \left(1+|x|^{2p}\right) ,$$

$$\mathbb{E}\left[\sup_{s\in[0,\rho_\varnothing]} |X^x_\varnothing(s) - X^{x'}_\varnothing(s)|^{2p}\right] \leq \bar{C}_p \int_0^\infty e^{\bar{C}_p t} \alpha e^{-\alpha t} dt \, |x-x'|^{2p} ,$$

Therefore, define $\alpha_p := \bar{C}_p + \delta$ for $\delta > 0$, from comparing the previous expression with the gamma distribution with parameters p and δ, we get (2.2.9)-(2.2.10), with $C_p := \frac{\alpha \bar{C}_p}{(\alpha - \bar{C}_p)^q}\Gamma(q)$ for $\alpha > \alpha_p$. Therefore, the property holds for the label $\varnothing$.

Suppose that (2.2.9)-(2.2.10) hold for all labels up to generation $n-1$ with $n \geq 1$. Fix $i \in \mathcal{I}$ with $i = i_1 \cdots i_n$ where $i_1, \cdots, i_n \in \mathbb{N}$. We notice that, given Ω_i, the process $\left(X^x_i(s)\right)_{s\in[0,\rho_i]}$ (resp. $\left(X^{x'}_i(s)\right)_{s\in[0,\rho_i]}$) is a diffusion process starting from $X^x_j(\rho_j)$ (resp. $X^x_j(\rho_j)$) and driven by B_i. From Proposition 2.2.10 $X^{x'}_j(\rho_j)$ (resp. $X^{x'}_j(\rho_j)$), B_i and ρ_i are independent. We can therefore apply the arguments used for the label $\varnothing$ and we get

$$\mathbb{E}\Big[\sup_{s\in[0,\rho_i]} \left|X^x_i(s)\right|^{2p} \Big| \Omega_i\Big] \leq C_p \left(1 + \mathbb{E}\Big[\left|X^x_j(\rho_j)\right|^{2p} \Big|\Omega_i\Big]\right) ,$$

$$\mathbb{E}\left[\sup_{s\in[0,\rho_i]} |X^x_i(s) - X^{x'}_i(s)|^{2p} \middle| \Omega_i\right] \leq C_p \mathbb{E}\left[|X^x_j(\rho_j) - X^{x'}_j(\rho_j)|^{2p} \middle| \Omega_i\right] .$$

We then use the identity $\Omega_i = \Omega_j \cap \{\nu_j \geq i_n\}$ together with Proposition 2.2.10 and we get

$$\mathbb{E}\Big[\left|X^x_j(\rho_j)\right|^{2p} \Big|\Omega_i\Big] = \mathbb{E}\Big[\left|X^x_j(\rho_j)\right|^{2p} \Big|\Omega_j\Big] ,$$

$$\mathbb{E}\left[|X^x_j(\rho_j) - X^{x'}_j(\rho_j)|^{2p} \middle| \Omega_i\right] = \mathbb{E}\left[|X^x_j(\rho_j) - X^{x'}_j(\rho_j)|^{2p} \middle| \Omega_j\right] .$$

We then get the result from the induction assumption.

□

We shall assume in the sequel that $\alpha > \alpha_4$. We now study the law of the shifted diffusion trees. The following result provides a conditional independence property also called branching property.

Theorem 2.2.7 (Branching property)**.** *For a stopping line $\tau = (\tau_i)_{i\in\mathcal{I}}$, given $\mathcal{F}_\tau$, the shifted diffusion trees $T_{i,\tau_i} \circ X^x$, $i \in L_\tau$ are independent follows the law of the diffusion tree starting from $X^x_i(\tau_i)$:*

$$\mathbb{E}\left[\prod_{i\in L_\tau} f_i(T_{i,\tau_i} \circ X^x) \middle| \mathcal{F}_\tau\right] = \prod_{i\in L_\tau} \mathbb{E}\left[f_i(X^{x_i})\right]\Big|_{x_i = X^x_i(\tau_i)} \tag{2.2.11}$$

for any family $(f_i)_{i\in\mathcal{I}}$ of non-negative $\mathcal{F}$-measurable random variables.

Proof. We define the processes $(X_i^{*x})_{i\in\mathcal{I}}$ where $X_\varnothing^{*x}$ is defined on Ω^* by

$$X_\varnothing^{*x}(0) = x \quad (2.2.12)$$
$$dX_i^{*x}(s) = b(X_i^{*x}(s))ds + \sigma(X_i^{*x}(s))dB_i^*(s)\ , \quad s \geq 0 \quad (2.2.13)$$

and X_i^{*x}, $i \neq \varnothing$, is defined on Ω^* by

$$X_i^{*x}(0) = X_j^{*x}(\rho_j^*) \quad (2.2.14)$$
$$dX_i^{*x}(s) = b(X_i^{*x}(s))ds + \sigma(X_i^{*x}(s))dB_i^*(s)\ , \quad s \in [0, \rho_i^*] \quad (2.2.15)$$

for $i \in \mathcal{I}$ where j is the parent of i.

We then have $X^x\big(\Phi(\omega^*)\big) = \Phi\big(X_i^{*x}(\omega^*)\big)$ for all $\omega^* \in \Omega^*$. Indeed, from [142, Theorem 10.4], X_i^x can be written as

$$X_i^x(s) = \Psi(X_j^x(\rho_j), B_i)(s)\ , \quad s \geq 0\ .$$

for some progressive function $\Psi :\ \mathbb{R}^d \times C^0(\mathbb{R}_+, \mathbb{R}^d) \to C^0(\mathbb{R}_+, \mathbb{R}^d)$, with j the parent of i. Still using [142, Theorem 10.4], we get that $\Psi(X_j^{*x}(\rho_j^*), B_i^*)$ is also solution to (2.2.14)-(2.2.15). We therefore get by an induction that $X^{*i} = \Psi(X_j^{*x}(\rho_j^*), B_i^*)$.

For $i \in \mathcal{I}$ and $s \in \mathbb{R}_+$, we define the translation map $T_{i,s}^* :\ \Omega^* \cap \{\rho_i^* \geq s\} \to \Omega^*$ by

$$B_\varnothing^*\big(T_{i,s}^*(\omega^*), t\big) = B_i^*\big(\omega^*, t+s\big) - B_i^*\big(\omega^*, s\big)\ , \quad (2.2.16)$$
$$B_j^*\big(T_{i,s}^*(\omega^*), t\big) = B_{ij}^*\big(\omega^*, t\big)\ ,\ j \neq \varnothing\ , \quad (2.2.17)$$
$$\rho_\varnothing^*\big(T_{i,s}^*(\omega^*)\big) = \rho_i^*(\omega^*) - s\ , \quad (2.2.18)$$
$$\rho_j^*\big(T_{i,s}^*(\omega^*)\big) = \rho_{ij}^*(\omega^*)\ ,\ j \neq \varnothing\ , \quad (2.2.19)$$
$$\nu_j^*\big(T_{i,s}^*(\omega^*)\big) = \nu_{ij}^*(\omega^*)\ , \quad (2.2.20)$$

for $j \in \mathcal{I}$ and $t \in \mathbb{R}_+$. Then the operators $T_{i,s}$ and $T_{i,s}^*$ are related by

$$T_{i,s} \circ \Phi = \Phi \circ T_{i,s}^* \quad \text{on} \quad \Phi^{-1}\big(\Omega_i \cap \{s \leq \rho_i\}\big) \quad (2.2.21)$$

for $i \in \mathcal{I}$ and $s \in \mathbb{R}_+$.

Fix now a stopping line $\tau = (\tau_i)_{i\in\mathcal{I}}$ and define the map $\tau^* = (\tau_i^*)_{i\in\mathcal{I}}$ by

$$\tau_i^* = \tau_i \circ \Phi \ \text{ on } \ \Phi^{-1}(\Omega_i)\ .$$

We also define the random set $L_{\tau^*}^*$ by

$$L_{\tau^*}^*(\omega^*) = \Big\{ i \in \pi^0(\Phi(\omega^*))\ :\ 0 \leq \tau_i^*(\omega^*) < \rho_i^*(\omega^*) \Big\}$$
$$= L_\tau(\Phi(\omega^*))$$

for $\omega^* \in \Omega^*$. In view of (2.2.16) to (2.2.20) , the maps $T_{i,\tau^*}^* \circ X^{*x}$, $i \in L_{\tau^*}^*$ are mutually independent given $\mathcal{F}_{\tau^*}^* = \Phi^{-1}(\mathcal{F}_\tau)$ and the law of $T_{i,\tau^*}^* \circ X^{*x}$ given $\mathcal{F}_{\tau^*}^*$ is given by

$$\mathcal{L}(T_{i,\tau^*}^* \circ X^{*x} | \mathcal{F}_{\tau^*}^*) = \mathcal{L}(X^{*x_i})\big|_{x_i = X_{\tau_i^*}^{*x}}\ .$$

Since $L^*_{\tau^*} \in \mathcal{F}^*_{\tau^*}$, we have

$$\mathbb{E}^*\left[\prod_{i\in B} f_i(T^*_{i,\tau^*}\circ X^{*x})\Big|\mathcal{F}^*_{\tau^*}\right] \;=\; \prod_{i\in B}\mathbb{E}^*\left[f_i(X^{*x_i}_i)\right]\Big|_{x_i=X^{*x}_i(\tau^*_i)}$$

on $\{B\subset L^*_{\tau^*}\}$ for any finite subset B of $\mathcal{I}$. Using (2.2.21) we get

$$\mathbb{E}^*\left[\prod_{i\in B} f_i(T_{i,\tau}\circ X^{x})\Big|\mathcal{F}_{\tau}\right] \;=\; \prod_{i\in B}\mathbb{E}^*\left[f_i(X^{x_i}_i)\right]\Big|_{x_i=X^{x}_i(\tau_i)}$$

on $\{B\subset L_\tau\}$ for any finite subset B of $\mathcal{I}$. Since L_τ is $\mathcal{F}_\tau$-measurable, we get the result. □

2.3 The optimal stopping problem

Prior to delving into the optimal stopping problem, we consider a modified version of the preceding scenario. Specifically, akin to the standard context, we incorporate an actualization component. This discount is applied to each branch, aligning with the type of cost function observed in [42] within the context of the optimal control setting. This augmentation involves the introduction of an extra temporal dimension into the previously introduced framework.

Let us now extend the dimension of the problem to $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$. We consider elements of this space to be denoted as $\tilde{x}=\binom{x}{y}$, with $x\in\mathbb{R}^d$ and $y\in\mathbb{R}$. We define now each particle $\tilde{X}^{\tilde{x}}_i=\begin{pmatrix}X^x_i\\ Y^y_i\end{pmatrix}$ to satisfy (2.2.4)-(2.2.7), with respect to $\left(\tilde{b},\tilde{\sigma}\right)$ defined as follows

$$\tilde{b}(\tilde{x})=\begin{pmatrix}b(x)\\ 1\end{pmatrix},\quad \tilde{\sigma}(\tilde{x})=\begin{pmatrix}\sigma(x)\\ 0\end{pmatrix},$$

with b and σ satisfying (2.2.8). It is clear that under these assumptions, we have that $Y^y_i(s)=S_i+s$. We fix now a function $g_i:\mathbb{R}^d\to\mathbb{R}_+$ for $i\in\mathcal{I}$, and we make the following assumption.

Assumption A6. *(i) The functions g_i, $i\in\mathcal{I}$, are non-negative and vanish uniformly in x as i goes to ∞, i.e., and*

$$\lim_{|i|\to+\infty}\sup_{x\in\mathbb{R}^d}\left|g_i(x)\right| \;=\; 0\,. \tag{2.3.22}$$

(ii) The functions g_i, $i\in\mathcal{I}$, are Lipschitz continuous uniformly in $i\in\mathcal{I}$, i.e., there exists a constant $L>0$ such that

$$|g_i(x)-g_i(x')| \;\le\; L|x-x'| \tag{2.3.23}$$

for all $i\in\mathcal{I}$ and $x,x'\in\mathbb{R}^d$.

The first assumption encodes a degradation of reward as we move too far away from the mother particle, both in generation and in number of children. This will allow us to have a system of differential equations indexed on a tree as in Chapter 1.

Fix now a constant $\gamma>0$. For the label $\varnothing$, we define the reward function $J_\varnothing$ by

$$J_\varnothing(x,\tau) \;=\; \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma Y^0_j(\tau_j)}g_j\big(X^x_j(\tau_j)\big)\right]=\mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma(S_j+\tau_j)}g_j\big(X^x_j(\tau_j)\big)\right]$$

for $x \in \mathbb{R}^d$, and any stopping line τ. Consequently, using the symmetry highlighted in the Theorem 2.2.7, we define the reward function starting for $i \in \mathcal{I}$ as follows

$$J_i(x,\tau) \quad = \quad \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma Y_j^0(\tau_j)} g_{ij}\big(X_j^x(\tau_j)\big)\right] = \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma(S_j+\tau_j)} g_j\big(X_j^x(\tau_j)\big)\right] \quad (2.3.24)$$

for $i \in \mathcal{I}$, $x \in \mathbb{R}^d$, and any stopping line τ. Let $v_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be the following value function

$$v_i(x) \quad = \quad \sup_{\tau\in\mathcal{SL}} J_i(x,\tau) \qquad (2.3.25)$$

for $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$. Our goal is to provide an analytic characterization of the family of functions $(v_i)_{i\in\mathcal{I}}$. We first state the basic properties of this family.

Proposition 2.3.12. *Suppose that Assumption A5 holds. There exist $\underline{\gamma} > 0$ such that for any $\gamma \geq \underline{\gamma}$, we have the following.*

(i) *The functions v_i are well defined and there exists $p \geq 1$ and a constant $C > 0$ such that*

$$|v_i(x)| \quad \leq \quad C\big(1+|x|^p\big), \qquad \text{for } i\in\mathcal{I},\ x\in\mathbb{R}^d. \qquad (2.3.26)$$

(ii) *There exists a constant $\bar{L} > 0$ such that*

$$|v_i(x)-v_i(x')| \quad \leq \quad \bar{L}|x-x'|C\left(1+|x|^p+|x'|^p\right), \qquad \text{for } i\in\mathcal{I},\ x,x'\in\mathbb{R}^d. \qquad (2.3.27)$$

(iii) *The functions v_i vanish uniformly in x as i goes to ∞, i.e.,*

$$\lim_{|i|\to+\infty}\sup_{x\in\mathbb{R}^d}\big|v_i(x)\big| \quad = \quad 0. \qquad (2.3.28)$$

Proof. We suppose $i = \varnothing$. The general case for $i \in \mathcal{I}$ is proven by renaming the functions g_j with g_{ij} for any $j \in \mathcal{I}$.

(i) Fix $x \in \mathbb{R}^d$ and any stopping line τ. Let N be an integer such that

$$\sup_{x\in\mathbb{R}^d}\big|g_i(x)\big| \quad \leq \quad \frac{1}{2} \qquad (2.3.29)$$

for $|i| \geq N$. Then, we have

$$\begin{aligned}\mathbb{E}\left[\Big|\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\Big|\right] \quad &= \quad \mathbb{E}\left[\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)}\big|g_i\big(X_i^x(\tau_i)\big)\big|\right]\\ &\leq \quad \mathbb{E}\left[\prod_{i\in L_\tau,|i|\leq N} e^{-\gamma(S_i+\tau_i)}\big|g_i\big(X_i^x(\tau_i)\big)\big|\right].\end{aligned}$$

We notice that

$$\#\{i\in L_\tau, |i|\leq N\} \quad \leq \quad \#\{i\in\mathcal{I}, |i|\leq N\} := \tilde{N} < \infty\,. \qquad (2.3.30)$$

Therefore using the inequality

$$\prod_{i=1}^{p} a_i \quad \leq \quad \sum_{i=1}^{p} a_i^p \ , \quad p \in \mathbb{N} \setminus \{0\} \ , \ a_1, \ldots, a_p \in \mathbb{R}_+ \ , \tag{2.3.31}$$

we get

$$\mathbb{E}\Big[\Big|\prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\Big|\Big] \quad \leq \quad \sum_{i \in \mathcal{I}, |i| \leq N} \mathbb{E}\Big[\big|g_i\big(X_i^x(\tau_i)\big)\big|^{\tilde{N}} \big|\Omega_i\Big] \mathbb{P}(\Omega_i)$$

Using Assumption A6 (ii), we get a constant C such that

$$\mathbb{E}\Big[\Big|\prod_{i \in L_\tau} g_i\big(X_i^x(\tau_i)\big)\Big|\Big] \quad \leq \quad C \sum_{i \in \mathcal{I}, |i| \leq N} \Big(1 + \mathbb{E}\Big[\big|X_i^x(\tau_i)\big|^{\tilde{N}} \big|\Omega_i\Big]\Big) \mathbb{P}(\Omega_i) \ .$$

From (2.2.9), we get a constant C' such that

$$\mathbb{E}\Big[\Big|\prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\Big|\Big] \quad \leq \quad C'\Big(1 + |x|^{\tilde{N}}\Big)$$

for all $x \in \mathbb{R}^d$. Therefore $J_\varnothing(x, \tau)$ is well defined for any $x \in \mathbb{R}^d$ and any stopping line τ and we get (2.3.26) with $p = \tilde{N}$.

(ii) Fix $x, x' \in \mathbb{R}^d$ and a stopping line τ. Then we have

$$|J_\varnothing(x,\tau) - J_\varnothing(x',\tau)| \quad \leq \quad \mathbb{E}\Bigg[\Bigg|\prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^x(\tau_i)\right) - \prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^{x'}(\tau_i)\right)\Bigg|\Bigg] \ .$$

We define

$$|\tau| \quad = \quad \max\{|i| \ : \ i \in L_\tau\} \ .$$

From Assumption A6 (i), we have

$$J_\varnothing(x,\tau) \quad = \quad J_\varnothing(x',\tau) \quad = \quad 0 \quad \text{ on } \quad \{|\tau| = +\infty\} \ .$$

We now work on $\{|\tau| < +\infty\}$. Using the inequality

$$\begin{aligned}\left|\prod_{i=1}^{p} a_i - \prod_{i=1}^{p} b_i\right| \quad &\leq \quad \sum_{i=1}^{p} |a_i - b_i| \prod_{j=1}^{i-1} a_i \prod_{j=i+1}^{p} b_i \\ &\leq \quad \sum_{i=1}^{p} |a_i - b_i| \prod_{\substack{j=1 \\ j \neq i}}^{p} a_i \vee b_i \ ,\end{aligned}$$

for $p \in \mathbb{N} \setminus \{0\}$ and $a_1, b_1, \ldots, a_p, b_p \in \mathbb{R}_+$, we have that

$$\begin{aligned}
&\left| \prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^x(\tau_i)\right) - \prod_{i \in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^{x'}(\tau_i)\right) \right| \\
\leq & \sum_{i \in L_\tau} \prod_{\substack{j \in L_\tau, \\ j \neq i, |j| \leq N}} e^{-\gamma(S_j+\tau_j)} \left(\left| g_i\left(X_j^x(\tau_j)\right) \right| \vee \left| g_i\left(X_j^{x'}(\tau_j)\right) \right| \right) e^{-\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right) \right|,
\end{aligned}$$

with N as in (2.3.29).

Taking expectation in the previous equation and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|J_\varnothing(x,\tau) - J_\varnothing(x',\tau)| \quad \leq \quad & \sum_{i \in \mathcal{I}} \mathbb{E}\left[\prod_{\substack{j \in L_\tau, \\ j \neq i, |j| \leq N}} e^{-2\gamma(S_j+\tau_j)} \left(\left| g_i\left(X_j^x(\tau_j)\right) \right|^2 \vee \left| g_i\left(X_j^{x'}(\tau_j)\right) \right|^2 \right) \right]^{1/2} \\
& \mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right) \right|^2 \mathbb{1}_{i \in L_\tau} \right]^{1/2}.
\end{aligned}$$

Using (2.3.31), there exists a constant $C > 0$ (which may change from line to line) such that

$$\begin{aligned}
\mathbb{E}\left[\prod_{\substack{j \in L_\tau, \\ j \neq i, |j| \leq N}} e^{-2\gamma(S_j+\tau_j)} \left(\left| g_i\left(X_j^x(\tau_j)\right) \right|^2 \vee \left| g_i\left(X_j^{x'}(\tau_j)\right) \right|^2 \right) \right] \quad & \leq \\
\mathbb{E}\left[\sum_{\substack{j \in L_\tau, \\ j \neq i, |j| \leq N}} e^{-2\gamma(S_j+\tau_j)\tilde{N}} \left(\left| g_i\left(X_j^x(\tau_j)\right) \right|^{2\tilde{N}} + \left| g_i\left(X_j^{x'}(\tau_j)\right) \right|^{2\tilde{N}} \right) \right] \quad & \leq \quad C(1 + |x|^{2\tilde{N}} + |x'|^{2\tilde{N}}),
\end{aligned}$$

where the last inequality derives from Assumption A6 (ii) and (2.2.9). We therefore get

$$\begin{aligned}
&|J_\varnothing(x,\tau) - J_\varnothing(x',\tau)| \quad \leq \\
&\qquad C\left(1 + |x|^{\tilde{N}} + |x'|^{\tilde{N}}\right) \sum_{i \in \mathcal{I}} \mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right) \right|^2 \mathbb{1}_{i \in L_\tau} \right]^{1/2}
\end{aligned}$$

for a constant $C > 0$ (which may change from line to line). Therefore, we obtain (2.3.27) if we prove that there exists $\underline{\gamma}$ such

$$\mathbb{E}\left[\sum_{i \in L_\tau} e^{-2\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right) \right|^2 \right] \quad \leq \quad C|x - x'|^2. \tag{2.3.32}$$

for any $\gamma \geq \underline{\gamma}$ and any stopping line τ.

Using (2.3.23), we have that the l.h.s. in (2.3.32) can be written as

$$\begin{aligned}
& \mathbb{E}\left[\sum_{i \in L_\tau} e^{-2\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right)\right|^2\right] \\
\leq \quad & L \sum_{i \in \mathcal{I}} \mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)} \left|X_i^x(\tau_i) - X_i^{x'}(\tau_i)\right|^2 \mathbb{1}_{i \in L_\tau}\right] \\
\leq \quad & L \sum_{i \in \mathcal{I}} \mathbb{E}\left[e^{-4\gamma S_i} \middle| \Omega_i\right]^{1/2} \mathbb{E}\left[\left|X_i^x(\tau_i) - X_i^{x'}(\tau_i)\right|^4 \middle| \Omega_i\right]^{1/2} \mathbb{P}\left(\Omega_i\right),
\end{aligned}$$

where, in the last inequality, we applied again Cauchy-Schwarz inequality and that $\{i \in L_\tau\} \subseteq \Omega_i$. For $i = i_1 \cdots i_n$, we have that

$$\mathbb{P}(\Omega_i) = \mathbb{P}\left(\nu_\varnothing \geq i_1, \ldots, \nu_{i_1 \cdots i_{n-1}} \geq i_n\right) = \left(\sum_{k \geq i_1} p_k\right) \cdots \left(\sum_{k \geq i_n} p_k\right) = \bar{p}_{i_1} \cdots \bar{p}_{i_n},$$

with $\bar{p}_\ell = \sum_{k \geq \ell} p_k$. Moreover, since S_i is the sum of n independent exponential random variable with parameter α, S_i is gamma-distributed with shape n and scale α. This means that its exponential moment is

$$\mathbb{E}\left[e^{-4\gamma S_i} \middle| \Omega_i\right] = \frac{\alpha^n}{(\alpha + 4\gamma)^n}.$$

Combining these two results with (2.2.10) for $p = 4$, we get for a constant $C > 0$ (which may change from line to line)

$$\begin{aligned}
& \mathbb{E}\left[\sum_{i \in L_\tau} e^{-2\gamma(S_i+\tau_i)} \left| g_i\left(X_i^x(\tau_i)\right) - g_i\left(X_i^{x'}(\tau_i)\right)\right|^2\right] \\
\leq \quad & L \sum_{n \geq 0} \frac{\alpha^{n/2}}{(\alpha + 4\gamma)^{n/2}} \left(\sum_{i_1 \geq 0} \bar{p}_{i_1}\right) \cdots \left(\sum_{i_n \geq 0} \bar{p}_{i_n}\right) (C_4)^{(n+1)/2} |x - x'|^2 \\
\leq \quad & C \left(\sum_{n \geq 0} \frac{\alpha^{n/2} M^n (C_4)^{n/2}}{(\alpha + 4\gamma)^{n/2}}\right) |x - x'|^2.
\end{aligned}$$

Therefore, for $\underline{\gamma} = \frac{\alpha|M^2 C_4 - 1|}{4} + \delta$ for $\delta > 0$, we get (2.3.32) and, a fortiori, (2.3.27).

(iii) The condition (2.3.28) is a clear consequence of Assumption (A6).

□

We shall assume in the sequel that $\gamma > \underline{\gamma}$ and we denote $\beta_s^i = e^{-\gamma(S_i+s)}$.

2.4 Dynamic programming principle

By leveraging the established regularity of the value function as proved in Proposition 2.3.12, we can now show the dynamic programming principle of our optimization problem. We prove this result by approximating the value functions using ε-optimal stopping lines. This technique is widely employed and can be found in [42] and [21]. It serves as an alternative approach, circumventing the need for measurable selection results, which can often be intricate and complex.

Theorem 2.4.8. *We have the following dynamic programming principle:*

$$v_i(x) = \sup_{\tau\in\mathcal{SL}} \mathbb{E}\Bigg[\prod_{j\in L_\theta\setminus D_\tau} \left(\beta^j_{\theta_j} v_{ij}\left(X^x_j(\theta_j)\right)\right)^{\mathbb{1}_{\{\theta_j\le\tau_j\}}} \prod_{j\in L_\tau\setminus D_\theta} \left(\beta^j_{\tau_j} g_{ij}\left(X^x_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta_j\}}}\Bigg], \tag{2.4.33}$$

for any $x\in\mathbb{R}^d$ and any stopping line θ.

Proof. Without loss of generality, we can suppose $i=\varnothing$. Fix a stopping line θ and denote $\bar v(x)$ on the right-hand side of 2.4.33. We first show that $v_\varnothing(x)\le \bar v(x)$. Fix a stopping line τ. The idea to follow is to divide the set L_τ between the particles that have already been stopped when looking at L_θ and the ones that have not yet been stopped. It is clear that

$$L_\tau = \left(L_\tau\setminus\left(D_\theta\cup L_\theta\right)\right)\cup L_\tau\cap\left(L_\theta\cup D_\theta\right).$$

Separating the stopping line τ between the branches that are stopped before and after θ, we get

$$\mathbb{E}\Bigg[\prod_{i\in L_\tau}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg] = \mathbb{E}\Bigg[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right) \prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg]. \tag{2.4.34}$$

Taking the conditional expectation given $\mathcal{F}_\theta$, we get

$$\mathbb{E}\Bigg[\prod_{i\in L_\tau}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg] = \mathbb{E}\Bigg[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right) \mathbb{E}\Bigg[\prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg|\mathcal{F}_\theta\Bigg]\Bigg].$$

We have by definitions that $\tau_i<\theta_i$ for $i\in L_\tau\setminus(D_\theta\cup L_\theta)$. Therefore we get

$$\mathbb{E}\Bigg[\prod_{i\in L_\tau}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg] = \mathbb{E}\Bigg[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\left(\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}} \mathbb{E}\Bigg[\prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\Bigg|\mathcal{F}_\theta\Bigg]\Bigg].$$

We then split the product on $L_\tau\cap L_\theta$ as follows

$$\prod_{i\in L_\tau\cap L_\theta}\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right) = \prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}} \prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i} g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i\le\tau_i\}}}.$$

This gives

$$\mathbb{E}\Bigg[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\Bigg] = \mathbb{E}\Bigg[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}}$$
$$\mathbb{E}\Bigg[\prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i\leq\tau_i\}}}\prod_{i\in L_\tau\cap D_\theta}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\Bigg|\mathcal{F}_\theta\Bigg]\Bigg].$$

We then notice that

$$L_\tau\cap D_\theta \quad = \quad \bigcup_{i\in L_\theta\setminus D_\tau}\{j\in L_\tau : i\prec j\}\,.$$

Using Theorem 2.2.7, we get that (2.4.34) can be rewritten as follows

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\big(X^x_i(\tau_i)\big)\right] \quad = \quad \mathbb{E}\Bigg[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta_i\}}}$$
$$\prod_{i\in L_\theta\setminus D_\tau}\left(\beta^i_{\theta_i}J_i\left(X^x_i(\theta_i),\tau^{i,\theta_i}\right)\right)^{\mathbb{1}_{\{\theta_i\leq\tau_i\}}}\Bigg],$$

with $\tau^{i,s}$ the stopping line defined on $\Omega\cap\{\tau_i\geq s\}$ as follows

$$\tau^{i,s}_\varnothing \quad = \quad (\tau_i-s)\mathbb{1}_{s\leq\tau_i<\rho_i}+(\rho_i-s)\mathbb{1}_{\tau_i=\rho_i}$$

and $\tau^{i,s}_j=\tau_{ij}$, for $j\in\mathcal{I}\setminus\{\varnothing\}$. From the definition of the value function v, we get

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\big(X^x_i(\tau_i)\big)\right] \quad \leq \quad \mathbb{E}\left[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta_i\}}}\prod_{i\in L_\theta\setminus D_\tau}\left(\beta^i_{\theta_i}v_i\left(X^x_i(\theta_i)\right)\right)^{\mathbb{1}_{\{\theta_i\leq\tau_i\}}}\right],$$

and

$$v_\varnothing(x) \quad \leq \quad \bar{v}(x)\,.$$

We now turn to the reverse inequality. Fix an open ball $B(x,r)$ for $r>0$ and a constant $\varepsilon\in(0,1)$. Define the stopping line θ^r by

$$\theta^r_\varnothing \quad = \quad \inf\big\{s\geq 0 : X^x_\varnothing(s)\notin B(x,r)\big\}\wedge\theta_\varnothing\wedge\rho_\varnothing,$$
$$\theta^r_i \quad = \quad \begin{cases}\inf\{s\geq 0 : X^x_i(s)\notin B(x,r)\}\wedge\theta_i\wedge\rho_i, & \text{if } \theta^r_j=\rho_j \text{ for any } j\prec i,\\ \theta^r_i=\rho_i, & \text{else.}\end{cases}$$

With this stopping line, consider the following function

$$\bar{v}_r(x) \quad = \quad \sup_{\tau\in\mathcal{SL}}\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_\tau}\left(\beta^i_{\theta^r_i}v_i\left(X^x_i(\theta^r_i)\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\tau_i\}}}\prod_{i\in L_\tau\setminus D_{\theta^r}}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta^r_i\}}}\right]$$

By definition, we can find a stopping line τ^ε such that

$$\bar{v}_r(x) \leq \mathbb{E}\Bigg[\prod_{i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}} \Big(\beta^i_{\theta^r_i} v_i(X^x_i(\theta^r_i))\Big)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}} \prod_{i\in L_{\tau^\varepsilon}\setminus D_{\theta^r}} \Big(\beta^i_{\tau^\varepsilon_i} g_i(X^x_i(\tau^\varepsilon_i))\Big)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\Bigg] + \varepsilon \,. \tag{2.4.35}$$

Consider now a partition $\{B_n\}_n$ of the closure of $B(x,r)$ and consider a sequence $\{x_n\}_n$ such that $x_n \in B_n$ for any $n \geq 0$. It is clear that we can find $\tau^{i,n} \in \mathcal{SL}$ such that

$$v_i(x_n) \leq J_i(x_n, \tau^{i,n}) + \varepsilon/3, \tag{2.4.36}$$

for any $i \in \mathcal{I}$. Moreover, the proof of (ii) in Proposition 2.3.12 shows that $J_i(\cdot,\tau)$ is a continuous function for any $i\in\mathcal{I}$ and any $\tau\in\mathcal{SL}$, and, from (2.3.22), have that J_i vanishes for i that tends to infinity. Combining this with the continuity of the value functions v_i for $i\in\mathcal{I}$, we have that the couple (x_n, B_n) can be chosen to satisfy the following

$$\max_{i\in\mathcal{I}}\big(|v_i(x)-v_i(x_n)| + |J_i(x,\tau^{i,n}) - J_i(x_n,\tau^{i,n})|\big) \leq \varepsilon/3, \quad \text{for } x\in B_n. \tag{2.4.37}$$

We next define the following family of random variables $\hat{\tau} = \{\hat{\tau}_i(x)\}_{x\in\mathbb{R}^d, i\in\mathcal{I}}$ by

$$\hat{\tau}_{i\ell}(x) = \tau^\varepsilon_{i\ell}$$

for $x\in\mathbb{R}^d$ and $i \in L_{\tau^\varepsilon}\setminus D_{\theta^r}$ such that $\tau^\varepsilon_i < \theta^r_i$ and $\ell\in\mathcal{I}$, and

$$\begin{aligned}
\hat{\tau}_i(x) &= \theta^r_i + \sum_{n\geq 0} \tau^{i,n}_{\varnothing}\mathbb{1}_{B_n}(x), && \text{for } x\in\mathbb{R}^d,\\
\hat{\tau}_{i\ell}(x) &= \sum_{n\geq 0}\tau^{i,n}_{\ell}\mathbb{1}_{B_n}(x) && \text{for } x\in\mathbb{R}^d,\ \ell\in\mathcal{I}\setminus\{\varnothing\}.
\end{aligned}$$

for $i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}$ such that $\theta^r_i \leq \tau^\varepsilon_i$. We observe that $(\hat{\tau}_i)_{i\in\mathcal{I}}$ defined by

$$\bar{\tau}_i = \hat{\tau}_i(X^x_i(\theta^r_i)) \,, \quad i\in\mathcal{I}$$

is a stopping line and from (2.4.36) and (2.4.37)

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}}\Big(\beta^i_{\theta^r_i} v_i\left(X^x_i(\theta^r_i)\right)\Big)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}} \prod_{i\in L_{\tau^\varepsilon}\setminus D_{\theta^r}}\Big(\beta^i_{\tau^\varepsilon_i} g_i\left(X^x_i(\tau^\varepsilon_i)\right)\Big)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right] \leq$$
$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_{\bar{\tau}}}\Big(\beta^i_{\theta^r_i}[J_i\left(X^x_i(\theta^r_i),(\bar{\tau}_{i\ell})_{\ell\in\mathcal{I}}\right)+\varepsilon]\Big)^{\mathbb{1}_{\{\theta^r_i\leq\bar{\tau}_i\}}} \prod_{i\in L_{\bar{\tau}}\setminus D_{\theta^r}}\left(\beta^i_{\bar{\tau}_i} g_i\left(X^x_i(\bar{\tau}_i)\right)\right)^{\mathbb{1}_{\{\bar{\tau}_i<\theta^r_i\}}}\right] .$$

From Assumption A6 (i), there exists $C>1$ is a constant such that

$$\sup_{i\in\mathcal{I}}\sup_{x\in\mathbb{R}^d}|g_i(x)| \leq C \,.$$

and $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^d} \big|g_i(x)\big| \quad \leq \quad \frac{1}{2} \,,$$

for $|i| \geq N$. We then have

$$|J_i(x,\tau)| \quad \leq \quad C^{\tilde{N}} \text{ if } |i| \; < \; N$$

and

$$|J_i(x,\tau)| \quad \leq \quad 1 \text{ if } |i| \; \geq \; N$$

for any $x \in \mathbb{R}^d$ and any stopping line τ, where $\tilde{N} = \#\{i \in \mathcal{I} \; : \; |i| \leq N\}$. We therefore get

$$\begin{aligned}
\prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i}[J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right)+\varepsilon] \quad &= \quad \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right) \\
&\qquad + \sum_{A\subsetneq L_{\theta^r}} \left(\prod_{i\in A} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right)\right) \varepsilon^{\#(L_{\theta^r}\setminus A)} \\
&\leq \quad \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right) + C^{\tilde{N}^2} \sum_{n=1}^{\#L_{\theta^r}} \varepsilon^n \\
&\leq \quad \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right) + C^{\tilde{N}^2} \frac{\varepsilon}{1-\varepsilon} \,,
\end{aligned}$$

where # stands for the cardinal. This means that the previous computation, together with (2.4.35), (2.4.36), and (2.4.37), implies that

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i\in L_{\theta^r}} \left(\beta^i_{\theta^r_i} v_i\left(X^x_i(\theta^r_i)\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}} \prod_{i\in L_\tau} \left(\beta^i_{\tau^\varepsilon_i} g_i\left(X^x_i(\tau^\varepsilon_i)\right)\right)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right] \\
\leq \quad &\mathbb{E}\left[\prod_{i\in L_{\theta^r}} \left(\beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}_i\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\bar{\tau}_i\}}} \prod_{i\in L_{\bar{\tau}}} \left(\beta^i_{\bar{\tau}_i} g_i(X^x_i(\bar{\tau}_i))\right)^{\mathbb{1}_{\{\bar{\tau}_i<\theta^r_i\}}}\right] + C^{\tilde{N}^2} \frac{\varepsilon}{1-\varepsilon} \,.
\end{aligned}$$

Using Theorem 2.2.7, we obtain

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}} \left(\beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar{\tau}\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\bar{\tau}_i\}}} \prod_{i\in L_{\bar{\tau}}} \left(\beta^i_{\bar{\tau}_i} g_i\left(X^x_i(\bar{\tau}_i)\right)\right)^{\mathbb{1}_{\{\bar{\tau}_i<\theta^r_i\}}}\right] \quad = \quad \mathbb{E}\left[\prod_{i\in L_{\bar{\tau}}} \beta^i_{\bar{\tau}_i} g_i\left(X^x_i(\bar{\tau}_i)\right)\right] .$$

Therefore, we achieve

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}} \left(\beta^i_{\theta^r_i} v_i\left(X^x_i(\theta^r_i)\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}} \prod_{i\in L_\tau} \left(\beta^i_{\tau^\varepsilon_i} g_i\left(X^x_i(\tau^\varepsilon_i)\right)\right)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right] \quad \leq \quad \mathbb{E}\left[\prod_{i\in L_{\bar{\tau}}} \beta^i_{\bar{\tau}_i} g_i\left(X^x_i(\bar{\tau}_i)\right)\right] + C^{\tilde{N}^2} \frac{\varepsilon}{1-\varepsilon} \,,$$

and

$$\bar{v}_r(x) \quad \leq \quad v_\varnothing(x) + \varepsilon + C^{\tilde{N}^2} \frac{\varepsilon}{1-\varepsilon} \,.$$

Since ε is arbitrarily chosen in $(0,1)$, we have $\bar{v}_r(x) \leq v_\varnothing(x)$. Letting r go to infinity, from Assumption A6, Proposition 2.2.11 and (2.3.26), we deduce that $\bar{v}(x) \leq v_\varnothing(x)$.

□

2.5 Dynamic programming equation

The value function associated with an optimal stopping problem is known to be the solution to an obstacle problem. To this purpose, we consider the following operator $\mathcal{L}$

$$\begin{array}{rcl} \mathcal{L} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R}^{\mathbb{N}} & \to & \mathbb{R} \\ \big(x, r, p, M, (r_\ell)_{\ell \in \mathbb{N}}\big) & \mapsto & \dfrac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)M\right) + b(x)^\top p + \alpha \displaystyle\sum_{k \geq 0} p_k \prod_{\ell=0}^{k-1} r_\ell - (\alpha+\gamma) r, \end{array}$$

with $\mathbb{S}^d$ being the set of symmetric matrices of dimension $d \times d$. We show in this section that the problem of stopping lines can be characterized by the following PDE

$$\min\left\{ -\mathcal{L}\left(x, v_i(x), Dv_i(x), D^2 v_i(x), \big(v_{i\ell}(x)\big)_{\ell\in\mathbb{N}}\right) \ ; \ v_i(x) - g_i(x) \right\} = 0, \tag{2.5.38}$$

for $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$. To simplify the notation, we will write $\mathcal{L}(i,v)(x)$ to denote $\mathcal{L}\Big(x, v_i(x), Dv_i(x), D^2 v_i(x), \big(v_{i\ell}(x)\big)_{\ell\in\mathbb{N}}\Big)$. Such a PDE shows a close connection to the underlying tree structure, having a coupling between the value function valued in i and in its direct offspring $i\ell$ for $\ell \geq 0$. Furthermore, when $r_\ell = r$ holds for all $\ell \in \mathbb{N}$, the convergence of the operator $\mathcal{L}$ is connected with the radius of convergence of the power series $\sum_{k\geq 0} p_k |x|^k$. This consideration leads us to introduce the following assumption.

Assumption A7. *The series $\sum_{k\geq 0} p_k |x|^k$ has infinite radius of convergence.*

We will prove that the value function (2.3.25) is a viscosity solution for (2.5.38). Before doing that, we introduce the notion of viscosity solution, like Definition 1.4.2.

Definition 2.5.4. *Let $u_i : \mathbb{R}^d \to \mathbb{R}_+$ be a continuous function for $i \in \mathcal{I}$.*
(i) $\{u_i\}_{i\in\mathcal{I}}$ is a viscosity supersolution *to* (2.5.38) *if, for $(i_0, x_0) \in \mathcal{I} \times \mathbb{R}^d$, $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$, and $\bar\varphi \in C^0(\mathbb{R}^d)$ such that φ_i is nonnegative for $i \in \mathcal{I}$,*

$$\sup_{i\in\mathcal{I}} \varphi_i(x) \quad \leq \quad \bar\varphi(x), \qquad \textit{for } x \in \mathbb{R}^d, \tag{2.5.39}$$

and

$$0 \ = \ \left(u_{i_0} - \varphi_{i_0}\right)(x_0) \quad = \quad \min_{\mathcal{I}\times\mathbb{R}^d} \left(u_\cdot - \varphi_\cdot\right),$$

we have

$$\min\left\{ -\mathcal{L}(i_0, \varphi_\cdot)(x_0) \ ; \ \varphi_{i_0}(x_0) - g_{i_0}(x_0) \right\} \geq 0,$$

(ii) $\{u_i\}_{i\in\mathcal{I}}$ is a viscosity subsolution *to* (2.5.38) *if, for $(i_0, x_0) \in \mathcal{I} \times \mathbb{R}^d$, $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$,*

and $\bar{\varphi} \in C^0(\mathbb{R}^d)$ *such that* φ_i *is nonnegative for* $i \in \mathcal{I}$, (2.5.39) *is satisfied, and*

$$0 \;=\; (u_{i_0} - \varphi_{i_0})\,(x_0) \;=\; \max_{\mathcal{I}\times\mathbb{R}^d} (u_{\cdot} - \varphi_{\cdot})\,,$$

we have

$$\min\left\{ -\mathcal{L}(i_0, \varphi_{\cdot})(x_0) \;;\; \varphi_{i_0}(x_0) - g_{i_0}(x_0) \right\} \le 0,$$

(iii) u *is a* viscosity solution *to* (2.5.38) *if it is both a viscosity sub and supersolution to* (2.5.38)*.*

Theorem 2.5.9. *Under Assumptions A5, A6, and A7 the value function* v *is a viscosity solution to* (2.5.38)*.*

Proof. We begin by proving the supersolution property. Fix $(i_0, x_0) \in \mathcal{I} \times \mathbb{R}^d$ and let $\varphi \in C^0(\mathbb{R}^d)$ and $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$ be such that

$$\sup_i |\varphi_i| \le \varphi \tag{2.5.40}$$

and

$$0 = (v_{i_0} - \varphi_{i_0})\,(x_0) = \min_{(i,x)\in\mathcal{I}\times\mathbb{R}^d} (v_i - \varphi_i)\,(x)\;. \tag{2.5.41}$$

Without loss of generality, we can assume this minimum to be strict in x once fixed i_0.

Consider, first, the following (trivial) stopping line τ^{triv}

$$\tau^{\text{triv}}_{\varnothing} := 0, \quad \text{and} \quad \tau^{\text{triv}}_j := \rho_j, \quad \text{for } j \in \mathcal{I}\backslash\{\varnothing\}.$$

Since τ^{triv} is admissible, combining it with (2.5.41), we get the inequality $v_{i_0}(x_0) = \varphi_{i_0}(x_0) \ge g_{i_0}(x_0)$.

Consider, now, the following stopping time

$$\bar{\theta}^h := \inf\left\{t > 0 : X^{x_0}_{\varnothing}(t) \notin B_1(x_0)\right\} \wedge h,$$

where $B_1(x_0)$ is the unit ball of $\mathbb{R}^d$ centred at x_0. With this stopping time, fix $h > 0$ and take the following stopping line θ^h

$$\begin{aligned}
\theta^h_{\varnothing} \;&:=\; \bar{\theta}^h \wedge \rho_{\varnothing},\\
\theta^h_{\ell} \;&:=\; \begin{cases} \rho_\ell & \text{if } \bar{\theta}^h < \rho_{\varnothing} \\ 0 & \text{else} \end{cases}\;, \quad \text{for } \ell \in \mathbb{N},\\
\theta^h_j \;&:=\; \rho_j, \quad \text{for } j \in \mathcal{I}\backslash\left(\{\varnothing\} \cup \mathbb{N}\right),
\end{aligned}$$

This stopping line stops at the exit time $\bar{\theta}^h$ or at the branching event $\rho_{\varnothing}$ if it arrives before $\bar{\theta}^h$. It follows from (2.4.33) applied with the stopping lines $\theta = \theta^h$ and $\tau = \theta^h$ that

$$v_{i_0}(x_0) \;\ge\; \mathbb{E}\left[\left(\beta^{\varnothing}_{\bar{\theta}^h} v_{i_0}\left(X^{x_0}_{\varnothing}\left(\bar{\theta}^h\right)\right)\right)\mathbb{1}_{\bar{\theta}^h < \rho_{\varnothing}} + \prod_{\ell=0}^{\nu_{\varnothing}-1}\left(\beta^{\varnothing}_{\rho_{\varnothing}} v_{i_0\ell}\left(X^{x_0}_{\ell}\left(0\right)\right)\right)\mathbb{1}_{\bar{\theta}^h \ge \rho_{\varnothing}}\right].$$

From the definition of $X^{x_0}_\ell(0)$ we get

$$v_{i_0}(x_0) \geq \mathbb{E}\left[\left(\beta^{\varnothing}_{\bar\theta^h} v_{i_0}\left(X^{x_0}_{\varnothing}\left(\bar\theta^h\right)\right)\right)\mathbb{1}_{\bar\theta^h<\rho_\varnothing} + \prod_{\ell=0}^{\nu_\varnothing-1}\left(\beta^{\varnothing}_{\rho_\varnothing} v_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(\rho_\varnothing\right)\right)\right)\mathbb{1}_{\bar\theta^h\geq\rho_\varnothing}\right].$$

Using (2.5.41), since the functions v_j and φ_j are positive for $j \in \mathcal{I}$, we have

$$\varphi_{i_0}(x_0) \geq \mathbb{E}\left[\left(\beta^{\varnothing}_{\bar\theta^h} \varphi_{i_0}\left(X^{x_0}_{\varnothing}(\bar\theta^h)\right)\right)\mathbb{1}_{\bar\theta^h<\rho_\varnothing} + \prod_{\ell=0}^{\nu_\varnothing}\left(\beta^{\varnothing}_{\rho_\varnothing-1} \varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(\rho_\varnothing\right)\right)\right)\mathbb{1}_{\bar\theta^h\geq\rho_\varnothing}\right].$$

From Proposition 2.2.10, we get

$$\begin{aligned}\varphi_{i_0}(x_0) &\geq \mathbb{E}\Bigg[\left(\beta^{\varnothing}_{\bar\theta^h} \varphi_{i_0}\left(X^{x_0}_{\varnothing}(\bar\theta^h)\right)\right)\mathbb{1}_{\bar\theta^h<\rho_\varnothing} \\ &\qquad + \sum_{k\geq1} p_k \prod_{\ell=0}^{k-1}\left(\beta^{\varnothing}_{\rho_\varnothing} \varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(\rho_\varnothing\right)\right)\right)\mathbb{1}_{\bar\theta^h\geq\rho_\varnothing}\Bigg] \\ &= \mathbb{E}\Bigg[\left(\beta^{\varnothing}_{\bar\theta^h\wedge\rho_\varnothing} \varphi_{i_0}\left(X^{x_0}_{\varnothing}(\bar\theta^h\wedge\rho_\varnothing)\right)\right) \\ &\qquad + \sum_{k\geq1} p_k \left(\prod_{\ell=0}^{k-1}\beta^{\varnothing}_{\rho_\varnothing} \varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(\rho_\varnothing\right)\right) - \beta^{\varnothing}_{\rho_\varnothing}\varphi_{i_0}\left(X^{x_0}_{\varnothing}\left(\rho_\varnothing\right)\right)\right)\mathbb{1}_{\bar\theta^h\geq\rho_\varnothing}\Bigg].\end{aligned}$$

Still using Proposition 2.2.10, we have

$$\begin{aligned}\varphi_{i_0}(x_0) &\geq \mathbb{E}\Bigg[\left(\beta^{\varnothing}_{\bar\theta^h\wedge\rho_\varnothing} \varphi_{i_0}\left(X^{x_0}_{\varnothing}(\bar\theta^h\wedge\rho_\varnothing)\right)\right) \\ &\qquad + \sum_{k\geq1} p_k \mathbb{E}\left[\int_0^{\bar\theta^h}\left(\prod_{\ell=0}^{k-1}\beta^{\varnothing}_{s} \varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(s\right)\right) - \beta^{\varnothing}_{s}\varphi_{i_0}\left(X^{x_0}_{\varnothing}\left(s\right)\right)\right)\alpha e^{-\alpha s}ds\right]\Bigg].\end{aligned}$$

Applying Itô's formula and Proposition 2.2.10, we get

$$\begin{aligned}0 \geq\ & \mathbb{E}\left[\int_0^{\bar\theta^h\wedge\rho_\varnothing}\beta^{\varnothing}_s\left(\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top D^2\varphi_{i_0}\right)\left(X^{x_0}_{\varnothing}(s)\right) + \left(b^\top D\varphi_{i_0}\right)\left(X^{x_0}_{\varnothing}(s)\right) - \gamma\varphi_{i_0}\left(X^{x_0}_{\varnothing}(s)\right)\right)ds\right] \\ & + \sum_{k\geq1} p_k \mathbb{E}\left[\int_0^{\bar\theta^h}\left(\prod_{\ell=0}^{k-1}\beta^{\varnothing}_{s} \varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}\left(s\right)\right) - \beta^{\varnothing}_{s}\varphi_{i_0}\left(X^{x_0}_{\varnothing}\left(s\right)\right)\right)\alpha e^{-\alpha s}ds\right]. \qquad (2.5.42)\end{aligned}$$

There exists $h_\omega > 0$, depending on ω, such that $\bar\theta_h = h$ and $\bar\theta_h < \rho_\varnothing$ for $h \leq h_\omega$. Then, dividing by $h > 0$ both sides of (2.5.42), we get from the mean value theorem and the dominated convergence theorem that $-\mathcal{L}\left(i_0, \varphi_.\right)(x_0) \geq 0$.

We turn now to the proof of the subsolution property. Fix $(i_0, x_0) \in \mathcal{I}\times\mathbb{R}^d$ and let $\varphi \in C^0(\mathbb{R}^d)$ and $\varphi_i \in C^2(\mathbb{R}^d)$ for $i \in \mathcal{I}$ be such that $\sup_i |\varphi_i| \leq \varphi$ and

$$0 = \left(v_{i_0} - \varphi_{i_0}\right)(x_0) = \max_{(i,x)\in\mathcal{I}\times\mathbb{R}^d}\left(v_i - \varphi_i\right)(x). \qquad (2.5.43)$$

Without loss of generality, we suppose that $i_0 = \varnothing$ and we take the maximum to be strict in x and that

$$\max_{(\ell,x)\in\mathbb{N}\times\mathbb{R}^d} (v_\ell - \varphi_\ell)(x) \quad = \quad -\delta \quad < \quad 0\,. \tag{2.5.44}$$

We argue by contradiction and assume that

$$2\eta := \min\left\{ -\mathcal{L}(\varnothing, \varphi.)(x_0)\ ;\ \varphi_\varnothing(x_0) - g_\varnothing(x_0) \right\} > 0.$$

Since all the functions in the previous inequality are continuous, we may find $\varepsilon > 0$ such that

$$-\mathcal{L}(\varnothing, e^{-\gamma s}(\varphi. - y))(x) \quad > \quad \eta\,, \tag{2.5.45}$$

$$(\varphi_\varnothing - g_\varnothing)(x) \quad > \quad \eta \tag{2.5.46}$$

for all $s, y \in [0, \varepsilon)$ and $x \in B_\varepsilon(x_0)$, with $B_\varepsilon(x_0)$ the open ball centred at x_0 with radius ε. Observe that, since x_0 is a strict maximizer, we have

$$-\zeta \quad = \quad \max_{\partial B_\varepsilon(x_0)} (v_\varnothing - \varphi_\varnothing)(x) < 0, \tag{2.5.47}$$

where $\partial B_\varepsilon(x_0)$ denotes the boundary of $B_\varepsilon(x_0)$. We now show that (2.5.45), (2.5.46), and (2.5.47) lead to a contradiction with (2.4.33). Define the stopping time $\bar\theta^\varepsilon$ by

$$\bar\theta^\varepsilon := \inf\left\{t > 0 : \left(t, X^{x_0}_\varnothing(t)\right) \notin [0,\varepsilon) \times B_\varepsilon(x_0)\right\}.$$

As for the supersolution property, we consider the stopping line θ^ε defined by

$$\begin{aligned}
\theta^\varepsilon_\varnothing &:= \bar\theta^\varepsilon \wedge \rho_\varnothing, \\
\theta^\varepsilon_\ell &:= \begin{cases} \rho_\ell & \text{if } \bar\theta^\varepsilon < \rho_\varnothing \\ 0 & \text{else} \end{cases}, \quad \text{for } \ell \in \mathbb{N}, \\
\theta^\varepsilon_j &:= \rho_j, \quad \text{for } j \in \mathcal{I}\setminus(\{\varnothing\} \cup \mathbb{N}),
\end{aligned}$$

This stopping line stops at the exit time $\bar\theta^\varepsilon$ or at the branching event $\rho_\varnothing$ if it arrives before $\bar\theta^\varepsilon$.

We next have from (2.5.44) and (2.5.47)

$$
\begin{aligned}
v_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j\le\tau_j\}}}\prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] &= \\
\varphi_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j\le\tau_j\}}}\prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] &= \\
\varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon<\rho_\varnothing\}}\left(\beta^\varnothing_{\bar\theta^\varepsilon} v_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon)\right)\right)^{\mathbb{1}_{\{\bar\theta^\varepsilon\le\tau_\varnothing\}}}\left(\beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right)^{\mathbb{1}_{\{\tau_\varnothing<\theta^\varepsilon_\varnothing\}}}\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing\ge\rho_\varnothing\}}\left(\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing} v_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)\right)\right] - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing<\rho_\varnothing\}}\beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right] &\ge \\
\varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon<\rho_\varnothing\}}\left(\beta^\varnothing_{\bar\theta^\varepsilon}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon)\right)-\zeta\right)\mathbb{1}_{\{\bar\theta^\varepsilon\le\tau_\varnothing\}} + \beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\mathbb{1}_{\{\tau_\varnothing<\theta^\varepsilon_\varnothing\}}\right)\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing\ge\rho_\varnothing\}}\left(\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\delta\right)\right)\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing<\rho_\varnothing\}}\beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x}_\varnothing(\tau_\varnothing)\right)\right] &\ge \\
\varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon<\rho_\varnothing\}}\beta^\varnothing_{\bar\theta^\varepsilon}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon)\right)-\zeta\right)\mathbb{1}_{\{\bar\theta^\varepsilon\le\tau_\varnothing\}}\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing\ge\rho_\varnothing\}}\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\delta\right)\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\tau_\varnothing<\bar\theta^\varepsilon\wedge\rho_\varnothing\}}\beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right] &
\end{aligned}
$$

for any stopping line τ. Using (2.5.46), we get

$$
\begin{aligned}
v_\varnothing(x) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j\le\tau_j\}}}\prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] &\ge \\
\varphi_\varnothing(x) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon<\rho_\varnothing\}}\beta^\varnothing_{\bar\theta^\varepsilon}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon)\right)-\zeta\right)\mathbb{1}_{\{\bar\theta^\varepsilon\le\tau_\varnothing\}}\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing\ge\rho_\varnothing\}}\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\delta\right)\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\tau_\varnothing<\bar\theta^\varepsilon\wedge\rho_\varnothing\}}\beta^\varnothing_{\tau_\varnothing}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)-\eta\right)\right] &\ge \\
\varphi_\varnothing(x) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\wedge\tau_\varnothing<\rho_\varnothing\}}\beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon\wedge\tau_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\right)\right] & \\
-\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\wedge\tau_\varnothing\ge\rho_\varnothing\}}\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\right)\right] &= \\
\varphi_\varnothing(x) - \mathbb{E}\left[\prod_{i\in\mathcal{V}_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}}\beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}\left(\varphi_i\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\right)\right] &.
\end{aligned}
$$

Applying Itô's formula we have

$$\varphi_\varnothing(x_0) - \mathbb{E}\left[\prod_{i\in\mathcal{V}_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}} \beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}\Big(\varphi_i\left(X^{x_0}_\varnothing(\rho_\varnothing)\right) - \zeta\wedge\eta\wedge\delta\wedge\varepsilon\Big)\right] \quad =$$

$$\zeta\wedge\eta\wedge\delta\wedge\varepsilon + \mathbb{E}\left[\int_0^{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing} -\mathcal{L}\big(\varnothing, \beta^\varnothing_s\left(\varphi_\cdot - \zeta\wedge\eta\wedge\delta\wedge\varepsilon\right)\big)(X^{x_0}_\varnothing(s))\right] \quad .$$

From (2.5.45) and the definition of $\bar\theta^\varepsilon$, we have

$$\mathbb{E}\left[\int_0^{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing} -\mathcal{L}(\varnothing, \varphi_\cdot - \zeta\wedge\eta\wedge\delta\wedge\varepsilon)(X^{x_0}_\varnothing(s))\right] \quad \geq \quad 0\,.$$

Therefore we get

$$v_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\Big(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\Big)^{\mathbb{1}_{\{\theta^\varepsilon_j\leq\tau_j\}}} \prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\Big(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\Big)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] \quad \geq \quad \zeta\wedge\eta\wedge\delta\wedge\varepsilon$$

for any stopping line τ. Since $\zeta\wedge\eta\wedge\delta\wedge\varepsilon > 0$, this contradicts (2.4.33).

□

We provide a strong comparison principle for the obstacle problem (2.5.38). The proof of this result is an extension of the usual comparison principle (see, *e.g.*, [138, 151]) with the use of some ideas from [42]. We consider an additional assumption and, for the sake of completeness, provide the complete proof. We recall that M is the mean of the branching mechanism, as in Assumption A5(ii).

Assumption A8. *(i) We have* $\gamma > \alpha(M-1)$.
(ii) The functions g_i, $i\in\mathcal{I}$, *are uniformly bounded, i.e.,*

$$\sup_{i\in\mathcal{I}}\sup_{x\in\mathbb{R}^d}\big|g_i(x)\big| \quad < \quad +\infty. \tag{2.5.48}$$

Mimicking the proof of point (i) of Proposition 2.3.12, we have that the value function is bounded as a consequence of the previous assumption. Therefore, we restrict to prove the following comparison theorem within the set of bounded viscosity solutions.

Prior to establishing the comparison principle, we present the subsequent preliminary lemma. We examine the alterations in the PDE (2.5.38) when a multiplicative penalization is applied to the viscosity solutions. In particular, take $\kappa > 0$, which will be fixed later, and define $\phi : \mathbb{R}^d \to \mathbb{R}$ as $\phi(x) = (|x|^2+1)^\kappa$, together with the following operator

$$\begin{aligned}\tilde{\mathcal{L}} : \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d\times\mathbb{R}^{\mathbb{N}} \quad &\to \quad \mathbb{R}\\ \big(x,r,p,M,(r_\ell)_{\ell\in\mathbb{N}}\big) \quad &\mapsto \quad \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)M\right) + \tilde b(x)^\top p + \alpha\sum_{k\geq 0} p_k\phi^{k-1}(x)\prod_{\ell=0}^{k-1} r_\ell - (\alpha+\tilde\gamma(x))r\,,\end{aligned}$$

with

$$\tilde{b}(x) = b(x) + \left(\frac{\sigma\sigma^\top D\phi}{\phi}\right)(x), \qquad \tilde{\gamma}(x) = \gamma - \left(\frac{b^\top D\phi}{\phi}\right)(x) - \frac{1}{2\phi(x)}\mathrm{Tr}\left(\sigma\sigma^\top D^2\phi\right)(x).$$

Lemma 2.5.3. *Let $\{u_i\}_{i\in\mathcal{I}}$ (resp. $\{v_i\}_{i\in\mathcal{I}}$) be a nonnegative lsc (resp. usc) viscosity supersolution (resp. subsolution) to* (2.5.38)*, satisfying* (2.3.26)*-*(2.3.28)*. Then, the functions $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ (resp. $\{\tilde{v}_i\}_{i\in\mathcal{I}}$) defined by*

$$\tilde{u}_i(x) = \frac{u_i(x)}{\phi(x)} \quad \left(\textit{resp. } \tilde{v}_i(x) = \frac{v_i(x)}{\phi(x)}\right), \quad \textit{for } x \in \mathbb{R}^d,$$

are nonnegative lsc (resp. usc) viscosity supersolution (resp. subsolution) to

$$\min\left\{-\tilde{\mathcal{L}}\left(x, \tilde{v}_i(x), D\tilde{v}_i(x), D^2\tilde{v}_i(x), \left(\tilde{v}_{i\ell}(x)\right)_{\ell\in\mathbb{N}}\right)\ ;\ \tilde{v}_i(x) - \tilde{g}_i(x)\right\} = 0, \tag{2.5.49}$$

with $\tilde{g}_i(x) = g_i(x)/\phi(x)$, for $(i,x)\in\mathcal{I}\times\mathbb{R}^d$.

Proof. We prove the supersolution case, the subsolution case is proven with the same techniques.

Fix $(i_0, x_0) \in \mathcal{I}\times\mathbb{R}^d$ and some test functions $\tilde{\varphi}_i \in C^2(\mathbb{R}^d)$, for $i\in\mathcal{I}$, and $\tilde{\varphi}\in C^0(\mathbb{R}^d)$ such that (2.5.39) and

$$0 \;=\; \left(\tilde{u}_{i_0} - \tilde{\varphi}_{i_0}\right)(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d}\left(\tilde{u}_\cdot - \tilde{\varphi}_\cdot\right)$$

are satisfied. Therefore, for $\varphi_i = \phi\tilde{\varphi}_i$ for $i\in\mathcal{I}$, we have

$$0 \;=\; \left(u_{i_0} - \varphi_{i_0}\right)(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d}\left(u_\cdot - \varphi_\cdot\right).$$

Moreover, the condition (2.5.39) is satisfied with respect to the function $\phi\bar{\varphi}$. Therefore, the functions φ_i for $i\in\mathcal{I}$ satisfies (2.5.38). Dividing this equation by the positive function ϕ and applying the product rule, we can see that the functions $\tilde{\varphi}_i$ for $i\in\mathcal{I}$ satisfy (2.5.49).

□

Theorem 2.5.10. *Let $\{u_i\}_{i\in\mathcal{I}}$ (resp. $\{v_i\}_{i\in\mathcal{I}}$) be a bounded nonnegative lsc (resp. usc) viscosity supersolution (resp. subsolution) to* (2.5.38)*, satisfying* (2.3.26)*-*(2.3.28)*. Then, under Assumptions A5, A6, A7 and A8, we have $u_i \le v_i$ for any $i\in\mathcal{I}$ on $\mathbb{R}^d$.*

Proof. From (2.3.28), there exists $N\in\mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^d} u_i(x) \vee \sup_{x\in\mathbb{R}^d} v_i(x) \quad \le \quad 1$$

for $i\in\mathcal{I}$ such that $|i|\ge N$. We proceed in two steps. We first show that $u_i\le v_i$ on $\mathbb{R}^d$ for any $i\in\mathcal{I}$ such that $|i|\ge N$. We then show this result for $|i|<N$.

Step 1. Denote by $\mathcal{I}_N$ the set $\{i\in\mathcal{I}\ :\ |i|\ge N\}$. We now prove that $u_i\le v_i$ on $\mathbb{R}^d$ for $i\in\mathcal{I}_N$.

We assume to the contrary that there exists $(z,j)\in\mathbb{R}^d\times\mathcal{I}_N$ such that

$$u_j(z) - v_j(z) \quad \ge \quad \delta, \tag{2.5.50}$$

for some $\delta>0$. Take $\tilde{u}_i = u_i/\phi$ (resp. $\tilde{v}_i = v_i/\phi$) for $i\in\mathcal{I}$. Since u_i and v_i are bounded, we have

$$\lim_{(i,x)\to\infty}(\tilde{u}_i+\tilde{v}_i)(x)=0. \tag{2.5.51}$$

This, together with (2.5.50), (2.3.28), and the fact that $\phi>0$, implies that there exists $(i_0,x_0)\in\mathcal{I}_N\times\mathbb{R}^d$ such that

$$\bar{M}_{0+}:=\sup_{(i,x)\in\mathcal{I}_N\times\mathbb{R}^d}\tilde{u}_i(x)-\tilde{v}_i(x) \quad = \quad \tilde{u}_{i_0}(x_0)-\tilde{v}_{i_0}(x_0) \quad \geq \quad \frac{\delta}{\phi(z)} \quad > \quad 0. \tag{2.5.52}$$

For $n\geq 1$, consider the following quantity

$$\bar{M}_n \quad = \quad \sup_{(i,x,y)\in\mathcal{I}_N\times\mathbb{R}^d\times\mathbb{R}^d}\tilde{u}_i(x)-\tilde{v}_i(y)-\frac{n}{2}|x-y|^2.$$

From (2.5.51), there exists (i_n,x_n,y_n) such that

$$\bar{M}_n \quad = \quad \tilde{u}_{i_n}(x_n)-\tilde{v}_{i_n}(y_n)-\frac{n}{2}|x_n-y_n|^2.$$

From the definition of N, taking $x=y$ in the previous supremum, we obtain

$$0 \quad < \quad \frac{\delta}{\phi(z)} \quad \leq \quad \bar{M}_{0+} \quad \leq \quad \bar{M}_n \quad \leq \quad 2. \tag{2.5.53}$$

This yields

$$\frac{n}{2}|x_n-y_n|^2 \quad \leq \quad 2. \tag{2.5.54}$$

From (2.5.53) and (2.5.51), we have, up to a sub-sequence, $i_n=i^*$, for some $i^*\in\mathcal{I}_N$ and all n, and, $(x_n,y_n)\to(x^*,y^*)$ as $n\to\infty$. From (2.5.54), we have

$$\lim_{n\to\infty}|x_n-y_n| \quad = \quad 0 \quad \text{and} \quad x^*=y^*.$$

Moreover, from (2.5.53), we obtain

$$\lim_{n\to\infty}\frac{n}{2}|x_n-y_n|^2 \quad = \quad 0.$$

Without loss of generality, we can take the maximization point in (2.5.52) to be (i^*,x^*), *i.e.*, $(i_0,x_0)=(i^*,x^*)$. Since $(x_n,y_n)\in\mathbb{R}^d\times\mathbb{R}^d$ is a maximizer of $\bar{M}_n$, from Assumption A8, we may apply Ishii's lemma (see, *e.g.*, [48, Theorem 8.3]) and Lemma 2.5.3. Therefore, there exist $A_n,B_n\in\mathbb{S}^d$ such that

$$\begin{aligned}
\min\Big\{-\tilde{\mathcal{L}}\Big(x_n,\tilde{u}_{i_0}(x_n),n(x_n-y_n),A_n,\big(\tilde{u}_{i_0\ell}(x_n)\big)_{\ell\in\mathbb{N}}\Big)\ ;\ \tilde{u}_{i_0}(x_n)-\tilde{g}_{i_0}(x_n)\Big\} &\leq 0,\\
\min\Big\{-\tilde{\mathcal{L}}\Big(y_n,\tilde{v}_{i_0}(y_n),n(x_n-y_n),B_n,\big(\tilde{v}_{i_0\ell}(y_n)\big)_{\ell\in\mathbb{N}}\Big)\ ;\ \tilde{v}_{i_0}(y_n)-\tilde{g}_{i_0}(y_n)\Big\} &\geq 0,
\end{aligned}$$

and

$$-3n\mathbb{I}_{2d} \quad \leq \quad \begin{pmatrix}A_n & 0\\ 0 & -B_n\end{pmatrix} \quad \leq \quad 3n\begin{pmatrix}\mathbb{I}_d & -\mathbb{I}_d\\ -\mathbb{I}_d & \mathbb{I}_d\end{pmatrix}.$$

If there exists a subsequence of $\{x_n\}_n$, still denoted $\{x_n\}_n$, such that $\tilde{u}_{i_0}(x_n) - \tilde{g}_{i_0}(x_n) \leq 0$, we get

$$[\tilde{u}_{i_0}(x_n) - \tilde{g}_{i_0}(x_n)] - [\tilde{v}_{i_0}(y_n) - \tilde{g}_{i_0}(y_n)] \quad \leq \quad 0,$$

for any n. This is, however, in contradiction with (2.5.53), the fact that $(x_n, y_n) \to (x_0, x_0)$ and the definition of (i_0, x_0). Therefore, we have

$$-\tilde{\mathcal{L}}\left(x_n, \tilde{u}_{i_0}(x_n), n(x_n - y_n), A_n, \left(\tilde{u}_{i_0\ell}(x_n)\right)_{\ell\in\mathbb{N}}\right) \quad \leq \quad 0, \tag{2.5.55}$$

$$-\tilde{\mathcal{L}}\left(y_n, \tilde{v}_{i_0}(y_n), n(x_n - y_n), B_n, \left(\tilde{v}_{i_0\ell}(y_n)\right)_{\ell\in\mathbb{N}}\right) \quad \geq \quad 0, \tag{2.5.56}$$

for n large enough.

Since $i_0 \in \mathcal{I}_N$, we have

$$\sup_{x\in\mathbb{R}^d} \left(u_{i_0\ell} \vee v_{i_0\ell}\right)(x) \quad \leq \quad 1$$

for all $\ell \geq 0$. This implies that

$$\begin{aligned}
&\left|\sum_{k\geq 0} p_k \phi^{k-1}(x) \prod_{\ell=0}^{k-1} \tilde{u}_{i_0\ell}(x_n) - \sum_{k\geq 0} p_k \phi^{k-1}(x) \prod_{\ell=0}^{k-1} \tilde{v}_{i_0\ell}(y_n)\right| \\
&\quad \leq \sum_{k\geq 0} p_k \sum_{\ell=0}^{k-1} \left(\prod_{\bar{\ell}=0}^{\ell-1} u_{i_0\bar{\ell}}(x_n)\right) |\tilde{u}_{i_0\ell}(x_n) - \tilde{v}_{i_0\ell}(y_n)| \left(\prod_{\bar{\ell}=\ell+1}^{k-1} v_{i_0\bar{\ell}}(y_n)\right) \\
&\quad \leq \sum_{k\geq 0} p_k \sum_{\ell=0}^{k-1} |\tilde{u}_{i_0\ell}(x_n) - \tilde{v}_{i_0\ell}(y_n)| \leq M \left(\tilde{u}_{i_0}(x_n) - \tilde{v}_{i_0}(y_n)\right),
\end{aligned}$$

where in the last inequality we used that (x_n, y_n) is a maximizer of $\bar{M}_n$, and (2.5.53). Since

$$\frac{D\phi(x)}{\phi(x)} = \frac{2\kappa x}{|x|^2 + 1},$$

$\tilde{b}$ is locally Lipschitz. Moreover, since

$$\frac{D^2\phi(x)}{\phi(x)} = 4\kappa(\kappa - 1)\frac{xx^\top}{(|x|^2 + 1)^2} + 2\kappa\frac{\mathbb{I}_d}{|x|^2 + 1},$$

we get that $\tilde{\gamma} - \gamma$ is equal to a bounded function in $\mathbb{R}^d$ multiplied by κ. Then, there exists κ small enough such that

$$\tilde{\gamma}(x) - \alpha(M - 1) \quad \geq \quad \frac{\gamma - \alpha(M - 1)}{2} \quad > \quad 0$$

for all $x \in \mathbb{R}^d$. This means that, from (2.5.55)-(2.5.56), we get

$$\begin{aligned}
&(\tilde{\gamma}(x_n) - \alpha(M - 1))\tilde{u}_{i_0}(x_n) - (\tilde{\gamma}(y_n) - \alpha(M - 1))\tilde{v}_{i_0}(y_n) \quad \leq \\
&\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\left(x_n - y_n\right) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) \qquad .
\end{aligned}$$

Sending n to infinity, from Assumption A8, we obtain

$$0 \geq (\tilde{\gamma}(x_0) - \alpha(M-1))\left(\tilde{u}_{i_0}(x_0) - \tilde{v}_{i_0}(x_0)\right).$$

From the choice of κ, the previous equation is in contradiction to (2.5.50).

Step 2. We now prove that $u_i \leq v_i$ on $\mathbb{R}^d$ for $|i| \leq N$, by a backward induction on $|i|$. From Step 1, the results hold for $i \in \mathcal{I}$ such that $|i| \leq N$. Fix $q \in \{0, \dots, N-1\}$ and suppose that $u_i \leq v_i$ on $\mathbb{R}^d$ for $|i| = q+1$.

Fix $i_0 \in \mathcal{I}$ such that $|i_0| = q$. As in Step 1, we argue by contradiction and suppose that there exists $z \in \mathbb{R}^d$ such that

$$u_{i_0}(z) - v_{i_0}(z) \quad \geq \quad \delta, \tag{2.5.57}$$

for some $\delta > 0$. Consider $\tilde{u}_i$ and $\tilde{v}_i$ as before, which still satisfy (2.5.51) from Assumption A8(ii). This assumption also entails that there exists a constant $C > 0$ such that $\tilde{u}_i(x) + \tilde{v}_i(x) \leq C$ for any $i \in \mathcal{I}, x \in \mathbb{R}^d$. As in (2.5.53), we get

$$0 < \delta\phi(z) \leq \bar{M}_{0+} \leq \bar{M}_n \leq C, \tag{2.5.58}$$

with $\bar{M}_{0+}$ and $\bar{M}_n$ defined as in Step 1. This yields

$$\frac{n}{2}|x_n - y_n|^2 \leq C. \tag{2.5.59}$$

Proceeding as in Step 1, we get (2.5.55)-(2.5.56) for n large enough. Since $|i_0\ell| \geq q+1$, we have from the inductive hypothesis $u_{i_0\ell} \leq v_{i_0\ell}$ (therefore $\tilde{u}_{i_0\ell} \leq \tilde{v}_{i_0\ell}$) for any $\ell \geq 0$. Combining (2.5.55)-(2.5.56) with the nonnegativity of the functions $\tilde{u}_i$ and $\tilde{v}_i$, and the previous inequalities, we have

$$\begin{aligned}
(\tilde{\gamma}(x_n) + \alpha)\tilde{u}_{i_0}(x_n) - (\tilde{\gamma}(y_n) + \alpha)\tilde{v}_{i_0}(y_n) \quad &\leq \\
\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\,(x_n - y_n) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) + \\
\alpha \sum_{k \geq 0} p_k \left(\phi^{-(k-1)}(x_n) \prod_{\ell=0}^{k-1} \tilde{u}_{i_0\ell}(x_n) - \phi^{-(k-1)}(y_n) \prod_{\ell=0}^{k-1} \tilde{v}_{i_0\ell}(y_n)\right) \quad &\leq \\
\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\,(x_n - y_n) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) + \\
\alpha \sum_{k \geq 0} p_k \left(\phi^{-(k-1)}(y_n) \prod_{\ell=0}^{k-1} \tilde{u}_{i_0\ell}(y_n) - \phi^{-(k-1)}(y_n) \prod_{\ell=0}^{k-1} \tilde{v}_{i_0\ell}(y_n)\right) \quad &.
\end{aligned}$$

Therefore, we get a contradiction to (2.5.57), as in Step 1, sending n to infinity. The results hold for i_0 and by induction, the results hold for all $i \in \mathcal{I}$.

□

As an immediate consequence of Theorems 2.5.9 and 2.5.10, we have the following characterization of the value function v.

Corollary 2.5.2. *Under Assumptions A5, A6, A7 and A8, v is the unique nonnegative bounded viscosity solution to* (2.5.38)*, satisfying* (2.3.28)*.*

Chapter 3

Relaxed formulation for Controlled Branching Diffusion Processes, Existence of an Optimal Control and HJB Equation

Outline of the current chapter

This chapter corresponds to the paper [128], which has been submitted for publication.

Abstract: The focus of this article is studying an optimal control problem for branching diffusion processes. Initially, we introduce the problem in its strong formulation and expand it to include linearly growing drifts. Then, we present a relaxed formulation that provides a suitable characterization based on martingale measures. Considering weak controls, we prove they are equivalent to strong controls in the relaxed setting, and establish the equivalence between the strong and relaxed problem, under a Filippov-type convexity condition. Furthermore, by defining control rules, we can restate the problem as the minimization of a lower semi-continuous function over a compact set, leading to the existence of optimal controls both for the relaxed problem and the strong one. Finally, with a useful embedding technique, we show that the value function solves a system of HJB equations, establishing a verification theorem. We then apply it to a linear-quadratic example and a kinetic one.

3.1 Introduction

The focus of this paper is on populations that are optimally controlled. Specifically, we aim to show the presence of a strong control for controlled branching diffusion processes and to describe the optimal dynamics.

The class of branching diffusion processes describes the evolution of particles, whose spatial movement is modeled by a SDE. Introduced in [94, 95, 96, 144], their study has been developed extensively, especially for their use in the probabilistic representation of semilinear PDEs (see, *e.g.*, [87]) and in the regularized unbalanced optimal transport (see, *e.g.*, [11]).

Several examples of optimal control for branching processes are discussed in the literature (see, *e.g.*, [42, 125, 152] and Chapter 1). They have been introduced in [152], wherein their modeling employs a topological sum of Euclidean space. The control, living within a compact space, solely affects the drift of spatial movement. The author permits each particle to potentially be influenced by any other living particle, without imposing any additional assumptions on the structure of these interactions. Moreover, the running cost yields a high degree of generality as well, leading to a correspondingly complex differential characterization. By selecting the cost function as the product of functions associated with the living particles at the terminal time, [125] employs controlled branching processes as a probabilistic tool to examine a specific group of parabolic Bellman equations. In this study, the control, still confined to a compact set, influences both drift and volatility. A Hamilton–Jacobi–Bellman (HJB) equation is identified, establishing that the value function represents its unique (viscosity) solution.

In [42], the author goes further in the analysis of this setting. Initially, the controlled processes are described as measure-valued processes. Using Ulam–Harris–Neveu labeling (see, *e.g.*, [10]) to describe the genealogy of the particles, the author introduces a label set that assists in defining the branching events. A set of Brownian motions and Poisson random measures, indexed by these labels, are used to provide a strong formulation for the controlled branching processes. This facilitates the well-posedness for dynamics where drift, volatility, branching rate, and branching mechanisms are not only controlled but also dependent on the position of each particle. While these coefficients are still assumed to be bounded, the control space is no longer necessarily compact. Since the dynamics are coupled only through the control, the product structure of the cost yields a branching property that converts the problem into a finite-dimensional one. A PDE characterization of the value function is then obtained, leveraging the differential properties of the Euclidean space where each single particle is defined. In Chapter 1, a similar approach is also employed. Here, the symmetry of the reward function is again used to establish a different branching property that allows for finite-dimensional rewriting.

This article expands on previous work on optimal control of branching diffusion processes. Firstly, we introduce a coupling between the particle dynamics vie the empirical measure of the population, similar to the interactions in mean field control literature. Secondly, we consider unbounded control space, and we allow the drift to have linear growth in both space and control while keeping the other coefficients bounded. We derive an HJB equation to characterize the value function, taking advantage of the homeomorphism between the topological sum of Euclidean spaces, as in [152], and the subset of finite measures, as in [42] and Chapter 1. This results in a verification theorem that we later rewrite as a (sub)martingale condition, similar to [137], to verify optimality. This brings us closer to the description of these processes as measure-valued and facilitates intuition for solving optimization problems, applying these results to a linear quadratic example and a kinetic one.

The first part of this paper addresses the issue of the existence of optimal controls. We follow the approach of [64] and [86], which involves a relaxed formulation of the problem. This formalism introduces different descriptions of the control problem, namely control rules and natural controls, allowing for greater flexibility and easier manipulation of the controlled dynamics. Proving that a control rule (resp. natural control) with a lower cost can be constructed from any relaxed control (resp. control rule), we establish the equivalence between strong and relaxed problems. Furthermore, we show that the cost function is lower semicontinuous for the control rule case, and, under some coercivity assumptions, we confine the search for minima to a compact set under a suitable topology. This rewrites the original optimization as the minimization of a lower semicontinuous function over a compact space, establishing the existence of optimal values and controls.

Similar methodology has been used in mean-field control theory (see, *e.g.*, [7, 107]) or branching populations dynamics (see, *e.g.*, [44]). Our approach differs from [44] as they make large use of the indexation with respect to the label set. Nonetheless, we use the topology introduced in this article, to apply it to measures with finite first-order moments.

The study of measure-valued processes in $\mathbb{R}^d$ has been ongoing since the late nineties. In seminal works such as [121, 140, 141], these processes were introduced as solutions to martingale problems. This strategy, detailed for the case of diffusion processes in [74], allows for a more abstract yet clearer manipulation of these objects. In [54], this point of view is applied to describe various dynamics, such as Fleming-Viot processes and superprocesses. This formulation for measure-valued processes produced a profiling literature and goes back to the eighties and nineties, see, *e.g.*, [53, 58, 60, 59, 61, 62, 73, 135, 134, 136]. This point of view provides useful convergence criteria and methods for characterizing their uniqueness in law, which will be extensively used in the remainder of the paper. In particular, the relaxed formulation of a control problem relies on the martingale problem formulation, as described in [70]. By exploiting the symmetry of the cost function with respect to the labeling, we can confine controls to an admissible class that preserves this symmetry. This restriction does not affect the problem's value function under mild assumptions, but it is crucial for defining relaxed controls, which, to the best of our knowledge, is the first of its kind. The control is seen as a probability measure of the action space that depends not only on time but also on space. We begin by presenting the connection between strong and relaxed controls through Dirac measures, identifying the class of weak controls. We prove their law uniqueness and use Doob's functional representation theorem to refer to the strong formulation. Under a Filippov-type convexity condition is satisfied (see, *e.g.*, [78]), any relaxed control can be associated with a weak one with lower cost. This gives the equivalence between the strong and relaxed characterizations and provides optimal strong control via the identification of weak and strong controls.

Finally, when attempting to optimize trajectories, we consider the concept of the kinetic energy of the system. This is the case of the Schrödinger bridge problem, as in [83], where one seeks to identify the random evolution (*i.e.*, a probability measure on path-space) that is closest to a prior Markov diffusion evolution in the relative entropy sense, while also satisfying certain initial and final marginals. It has been noted that this problem can be framed as a stochastic control problem, see, *e.g.*, [40, 41, 51, 52], where the kinetic energy plays a fundamental role in the cost function. Continuing along this line of reasoning, we present an example involving a comparable cost function and proceed to solve it with the help of the verification theorem.

The rest of the paper is structured as follows. In Section 3.2, we provide an introduction to the setting and the strong formulation for controlled branching processes. The control problem is defined and its well-posedness is proven. In Section 3.3, we introduce the relaxed formulation, presenting equivalent representations and characterizing them using martingale measures. Section 3.4 establishes the equivalence between the relaxed and strong formulations under a Filippov-type convexity condition. We introduce natural controls in this setting and show that we can restrict the problem to this class by conditioning on measures. Then, we compare the embedding of strong controls with weak ones and show their equivalence via uniqueness in law for these objects and Doob's functional representation theorem. Section 3.5 introduces the set of control rules and uses it to prove the lower semicontinuity of the cost functions in this set. Here, we show there exists a minimal solution to the strong control problem, after restricting to a compact set found using the coercivity assumption of the cost. Finally, in Section 3.6, we present the system of HJB equations and use it to solve a linear quadratic example and a kinetic one.

3.2 The control problem

3.2.1 The set of measures

For a Polish space (E, d) with $\mathcal{B}(E)$ its Borelian σ-field, we write $C_b(E)$ (resp. $C_0(E)$) for the subset of the continuous functions that are bounded (resp. that vanish at infinity), and $M(E)$ (resp. $\mathcal{P}(E)$) for the set of Borel positive finite measures (resp. probability measures) on E. We equip $M(E)$ with weak* topology, *i.e.*, the weakest topology that makes continuous the maps $M(E) \ni \lambda \mapsto \int_E \varphi(x)\lambda(dx)$ for any $\varphi \in C_b(\mathbb{R}^d)$. We denote $\langle \varphi, \lambda \rangle = \int_E \varphi(x)\lambda(dx)$ for $\lambda \in M(E)$ and $\varphi \in C_b(E)$.

Denote also by $M^p(E)$ the subspace of measures with finite p-th moment for $p \geq 1$, *i.e.*, the collection of all $\lambda \in M(E)$ such that $\int_E d(x, x_0)^p \lambda(dx) < \infty$ for some $x_0 \in E$. The weak* topology can be metrized in $M^p(E)$ by the Wasserstein type metric $\mathrm{d}_{p,E}$, as introduced in [44, Appendix B]. This means that, if ∂ is a cemetery point, we consider first $\bar{E}$ the enlarged space $\bar{E} := E \cup \{\partial\}$. Defining $d(x, \partial) := d(x, x_0) + 1$, we have that $(\bar{E}, d)$ is Polish. On the space

$$M^p_m(\bar{E}) := \{\lambda \in M^p(\bar{E}) : \lambda(\bar{E}) = m\},$$

consider the Wasserstein distance as follows

$$\mathrm{d}_{p,E,m}(\lambda, \lambda') = \left(\inf_{\pi \in \Pi(\lambda, \lambda')} \int_{\bar{E} \times \bar{E}} d(x, y)^p \pi(dx, dy) \right)^{1/p}, \quad \text{for } \lambda, \lambda' \in M^p_m(\bar{E}),$$

where $\Pi(\lambda, \lambda')$ denotes the collection of all non-negative measures on $\bar{E} \times \bar{E}$ with marginals λ and λ'. The distance $\mathrm{d}_{p,E}$ on $M^p(E)$ is defined as

$$\mathrm{d}_{p,E}(\lambda, \lambda') = \mathrm{d}_{p,E,m}\left(\bar{\lambda}_m, \bar{\lambda}'_m\right), \quad \text{for } \lambda, \lambda' \in M^p_m(E),$$

where $m \geq \lambda(E) \vee \lambda'(E)$ and

$$\bar{\lambda}_m(\cdot) := \lambda(\cdot \cap E) + (m - \lambda(E))\delta_\partial(\cdot), \bar{\lambda}'_m(\cdot) := \lambda'(\cdot \cap E) + (m - \lambda'(E))\delta_\partial(\cdot).$$

As proven in [44, Lemma B.1], this definition does not depend on the choice of m. Moreover, for some $x_0 \in E$, we have the natural bound

$$\mathrm{d}^p_{p,E}(\lambda, \delta_{x_0}) \leq \int_E d(x, x_0)^p \lambda(dx) + \langle 1, \lambda \rangle^p, \qquad \text{for } \lambda \in M^p(E). \tag{3.2.1}$$

We can remark that all the results in [44, Appendix B], about the convergence under $\mathrm{d}_{1,E}$, can be directly generalized for $\mathrm{d}_{p,E}$.

Finally, we write $\mathcal{N}[E]$ for the space of atomic measures in E, *i.e.*,

$$\mathcal{N}[E] := \left\{ \sum_{i=1}^m \delta_{x_i} \;:\; m \in \mathbb{N}, x_i \in E \text{ for } i \leq m \right\},$$

a weakly* closed subset of $M(E)$. In particular, we remark that $\mathcal{N}[\mathbb{R}^d]$ is also a closed set of $M^p(\mathbb{R}^d)$ with respect to the distance $\mathrm{d}_{p,E}$. This is due to the fact that $\mathcal{N}[\mathbb{R}^d]$ is weakly*-closed and, from [44, Lemma B.2], convergence in $M^1(\mathbb{R}^d)$ entails weak*-convergence to some $\lambda \in \mathcal{N}[\mathbb{R}^d] \subseteq M^1(\mathbb{R}^d)$.

Remark 3.2.4. *Each vector $\vec{x}^m = (x_1, \dots, x_m) \in \mathbb{R}^{dm}$ can be embedded in $\mathcal{N}[\mathbb{R}^d]$ as $\iota(\vec{x}^m) := \sum_{i=1}^m \delta_{x_i}$. Fix $\vec{x}^m, \vec{y}^m \in \mathbb{R}^{dm}$. We use the characterisation of the distance $d_{1,E}$ of [44, Lemma B.1] and obtain*

$$d_{1,E}\left(\iota(\vec{x}^m), \iota(\vec{y}^m)\right) = \sup_{\varphi \in Lip_1^0(\mathbb{R}^d)} \sum_{i=1}^m |\varphi(x_i) - \varphi(y_i)| \leq \sum_{i=1}^m |x_i - y_i| = |\vec{x}^m - \vec{y}^m|.$$

where $Lip_1^0(\mathbb{R}^d)$ denote the collection of all functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ with Lipschitz constant smaller or equal to 1 and such that $\varphi(0) = 0$.

3.2.2 Strong formulation

Fix a finite time horizon $T > 0$. Let $\mathbf{D}^d = \mathbb{D}([0,T]; M^1(\mathbb{R}^d))$ be the set of càdlàg functions (right continuous with left limits) from $[0,T]$ to $M^1(\mathbb{R}^d)$. We endow this space with Skorohod metric $d_{\mathbf{D}^d}$ associated with the metric $\mathrm{d}_{\mathbb{R}^d}$, which makes it complete (see, *e.g.*, [20]). For $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$, $\mathbb{P}_t \in \mathcal{P}(M^1(\mathbb{R}^d))$ denotes the time-t marginal of $\mathbb{P}$, *i.e.*, the image of $\mathbb{P}$ under the map $\mathbf{D}^d \ni \mu \mapsto \mu_t \in M^1(\mathbb{R}^d)$.

Assumptions We are given dimensions $d, d' \in \mathbb{N}$, a closed subset A of $\mathbb{R}^m$ representing the set of actions, and the following continuous functions

$$(b, \sigma, \gamma, p_k) \quad : \quad \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A \to \mathbb{R}^d \times \mathbb{R}^{d \times d'} \times \mathbb{R}_+ \times [0,1]$$

for $k \geq 0$, such that $\sum_{k \geq 0} p_k(x, \lambda, a) = 1$ for any $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. Assume that b and σ are Lipschitz continuous in (x, λ), *i.e.*, there exists $L > 0$ such that

$$|b(x, \lambda, a) - b(x', \lambda', a)| + |\sigma(x, \lambda, a) - \sigma(x', \lambda', a)| \leq L(|x - x'| + \mathrm{d}_{\mathbb{R}^d}(\lambda, \lambda')), \tag{3.2.2}$$

for any $x, x' \in \mathbb{R}^d$, $\lambda, \lambda' \in M^1(\mathbb{R}^d)$, and $a \in A$. Suppose also that σ and γ are uniformly bounded, and b has linear growth in (x, a) while bounded in λ, *i.e.*, there exists $C_\sigma, C_\gamma, C_b > 0$ such that

$$|b(x, \lambda, a)| \leq C_b(1 + |x| + |a|), \qquad |\sigma(x, \lambda, a)| \leq C_\sigma, \qquad \gamma(x, \lambda, a) \leq C_\gamma, \tag{3.2.3}$$

for $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. Let Φ be the generating function of $(p_k)_k$, *i.e.*,

$$\Phi(s, x, \lambda, a) = \sum_{k=0}^{\infty} p_k(x, \lambda, a) s^k, \quad \text{for } (s, x, \lambda, a) \in [0, 1] \times \mathbb{R}^d \times M^1(E) \times A.$$

Assume that the first and second order moments related to $(p_k)_k$ are uniformly bounded, *i.e.*, there exist two constants $C_\Phi^1, C_\Phi^2 > 0$ such that

$$\begin{aligned} \partial_s \Phi(1, x, \lambda, a) &= \sum_{k \geq 1} k p_k(x, \lambda, a) \leq C_\Phi^1, \\ \partial_{ss}^2 \Phi(1, x, \lambda, a) &= \sum_{k \geq 1} k(k-1) p_k(x, \lambda, a) \leq C_\Phi^2, \end{aligned} \tag{3.2.4}$$

for any $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. The generalization to time-dependent coefficients is straightforward. We do not address it explicitly not to make the notation heavier. We will make use of this setting in Section 3.6.3.

Strong controls We consider the set of labels $\mathcal{I} = \{\varnothing\} \cup \bigcup_{n=1}^{+\infty} \mathbb{N}^n$ and use Ulam–Harris–Neveu labeling to consider the genealogy of the particles. Denote by $\varnothing$ the mother particle, and $i = i_1 \cdots i_n$ the multi-integer $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $n \geq 1$. For $i = i_1 \cdots i_n \in \mathbb{N}^n$ and $j = j_1 \cdots j_m \in \mathbb{N}^m$, we define their concatenation is $ij \in \mathbb{N}^{n+m}$ by $ij = i_1 \cdots i_n j_1 \cdots j_m$, and extend it to the entire $\mathcal{I}$ by $\varnothing i = i \varnothing = i$ for all $i \in \mathcal{I}$. When a particle $i = i_1 \cdots i_n \in \mathbb{N}^n$ gives birth to k particles, the off-springs are labelled $i0, \ldots, i(k-1)$.

Let $\left(\Omega^{\mathfrak{s}}, \mathbb{F}^{\mathfrak{s}} = \{\mathcal{F}_t^{\mathfrak{s}}\}_{t \geq 0}, \mathcal{F}^{\mathfrak{s}}, \mathbb{P}^{\mathfrak{s}}\right)$ be a filtered probability space satisfying the usual conditions. Suppose that this space supports two independent families $\{W^i\}_{i \in \mathcal{I}}$ and $\{Q^i\}_{i \in \mathcal{I}}$ of mutually independent processes. Let W^i be a d'-dimensional Wiener processes, and $Q^i(dsdz)$ a Poisson random measure on $[0, T] \times \mathbb{R}_+$ with intensity measure $dsdz$.

Definition 3.2.5 (Standard strong control)**.** *We say that* $\beta = (\beta^i)_{i \in \mathcal{I}}$ *is a* standard strong control *if* β *is an* $\mathbb{F}^{\mathfrak{s}}$*-predictable* $A^{\mathcal{I}}$*-valued process, such that*

$$\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}} \left[\sup_{i \in \mathcal{I}} \int_t^T |\beta_s^i|^2 ds \right] < \infty. \tag{3.2.5}$$

Fix a standard control $\beta = (\beta^i)_{i \in \mathcal{I}}$. We describe the *controlled branching diffusion process* ξ^β as the measure-valued process

$$\xi_t^\beta = \sum_{i \in V_t} \delta_{Y_t^{i,\beta}} \ ,$$

where $Y_t^{i,\beta}$ is the position of the member with label $i \in \mathcal{I}$, and V_t the set of alive particles at time t. This process takes values in $\mathcal{N}[\mathbb{R}^d]$ and the behaviour of each alive particle i is characterized by the following three properties:

– *Spatial motion*: during its lifetime, it moves in $\mathbb{R}^d$ according to the following stochastic differential equation

$$dY_s^{i,\beta} = b\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) ds + \sigma\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) dW_s \ ;$$

– *Branching rate* γ: given a position $Y_s^{i,\beta}$ at time s, the probability it dies in the time interval $[s, s+\delta s)$ is $\gamma\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right)\delta s + o(\delta s)$.

– *Branching mechanism*: when it dies at a time s, it leaves behind (at the location where it died) a random number of offspring with probability $\left(p_k\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right)\right)_{k\in\mathbb{N}}$.

If the control is constant, *i.e.*, we are in the uncontrolled setting, conditionally on time and place of birth, the offspring evolve independently of each other in the same way as their parent.

Let L be the generator (associated with the spatial motion of each particle) defined on $\varphi \in C_b^2(\mathbb{R}^d)$ as

$$L\varphi(x,\lambda,a) \quad = \quad b(x,\lambda,a)^\top D\varphi(x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,\lambda,a)D^2\varphi(x)\right) \ ,$$

where D and D^2 denote gradient and Hessian. The representation of previous properties is given by the following SDE

$$\begin{aligned}\langle\varphi,\xi_s^\beta\rangle \quad &= \quad \langle\varphi,\xi_t^\beta\rangle + \int_t^s \sum_{i\in V_u} D\varphi(Y_u^{i,\beta})^\top \sigma\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) dB_u^i + \int_t^s \sum_{i\in V_u} L\varphi\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) du \\ &\quad + \int_{(t,s]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 0} (k-1)\varphi(Y_u^{i,\beta}) \mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z) Q^i(dudz) \ ,\end{aligned} \tag{3.2.6}$$

with

$$I_k(x,\lambda,a) = \left[\gamma(x,\lambda,a)\sum_{\ell=0}^{k-1} p_\ell(x,\lambda,a), \gamma(x,\lambda,a)\sum_{\ell=0}^{k} p_\ell(x,\lambda,a)\right) \ ,$$

for all $(x,\lambda,a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$, $k \geq 0$, with the value of an empty sum being zero by convention. Notice that $(I_k(x,\lambda,a))_{k\in\mathbb{N}}$ forms a partition of the interval $[0, \gamma(x,\lambda,a))$.

Existence of branching processes and moment estimates

We aim to show the existence of controlled branching diffusion processes for any standard strong control and giving bounds on their moments. These two aspects are proved in the following two propositions, adapting [42, Proposition 2.1] to our context.

Proposition 3.2.13. *Let $t \in [0,T]$, $\lambda := \sum_{i\in V} \in \mathcal{N}[\mathbb{R}^d]$ with $V \subseteq \mathcal{I}$ finite, and β be a standard strong control. There exists a unique (up to indistinguishability) càdlàg and adapted process $\left(\xi_s^\beta\right)_{s\geq t}$ satisfying (3.2.6) such that $\xi_t^\beta = \lambda$. In addition, there exists a constant $C > 0$ depending* *only on T and on the coefficients b, σ, γ and $(p_k)_k$ such that*

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,t+h]}|V_u|\right] \leq \langle 1,\lambda\rangle e^{C_\gamma C^1_\Phi h}, \tag{3.2.7}$$

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,t+h]}|V_u|^2\right] \leq \langle 1,\lambda\rangle e^{C_\gamma (C^1_\Phi + C^2_\Phi) h}, \tag{3.2.8}$$

$$\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}\sum_{i\in V_u}|\beta^i_u|du\right] \leq C, \tag{3.2.9}$$

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,t+h]}\sum_{i\in V_u}\left|Y^{i,\beta}_u\right|\right] \leq C\Bigg(\sum_{i\in V}|x^i| + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}|V_u|du\right] + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}\sum_{i\in V_u}\left|\beta^i_u\right|du\right]\Bigg), \tag{3.2.10}$$

for any $h>0$, where $|V|$ denotes the cardinality of $V\subseteq\mathcal{I}$.

Proof. Fix $\left(t,\lambda=\sum_{i\in V}\delta_{x^i}\right)\in\mathbb{R}_+\times\mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. Using induction, we build the branching events of the population. We later show that such a process satisfies (3.2.6) and is well-defined. From (3.2.1), we have that (3.2.7) and (3.2.10) entail well-posedness for the process ξ^β.

Define by induction an increasing sequence of stopping time $(\tau_k)_{k\in\mathbb{N}}$, a sequence of random variables $(V_k)_{k\in\mathbb{N}}$ valued in the set of finite subsets of $\mathcal{I}$ and a sequence of processes $\left(\left(Y^{i,\beta}_s\right)_{s\in[\tau_{k-1},\tau_k)}, i\in V_k\right)_{k\in\mathbb{N}}$ such that

$$\xi^\beta_s = \sum_{k\geq 1}\mathbb{1}_{\tau_{k-1}\leq s<\tau_k}\sum_{i\in V_k}\delta_{Y^{i,\beta}_s}.$$

We set $\tau_0=t$, $V_0=V$, and $Y^{i,\beta}_t := x^i$ for all $i\in V$. Then, given τ_{k-1} and V_{k-1}, define τ_k as

$$\tau_k = \inf\left\{s\in(\tau_{k-1},T] : \exists i\in V_{k-1},\ Q^i((\tau_{k-1},s]\times[0,C_\gamma])=1\right\}.$$

Define $\mathcal{Y}^k$, $\mathfrak{b}^k(\mathcal{Y}^k,\beta_s)$, $\Sigma^k(\mathcal{Y}^k,\beta_s)$, and $\mathcal{W}^k$, as

$$\mathcal{Y}^k_s := \begin{pmatrix} Y^{i_1,\beta}_s \\ \vdots \\ Y^{i_{|V_{k-1}|},\beta}_s\end{pmatrix}, \quad \mathfrak{b}^k(\mathcal{Y}^k_s,\beta_s) := \begin{pmatrix} b\left(Y^{i_1,\beta}_s, \sum_{i\in V_{k-1}}\delta_{Y^{i,\beta}_s}, \beta^{i_1}_s\right) \\ \vdots \\ b\left(Y^{i_{|V_{k-1}|},\beta}_s, \sum_{i\in V_{k-1}}\delta_{Y^{i,\beta}_s}, \beta^{i_{|V_{k-1}|}}_s\right)\end{pmatrix},$$

$$\Sigma^k(\mathcal{Y}^k_s,\beta_s) := \begin{pmatrix} \sigma\left(Y^{i_1,\beta}_s, \sum_{i\in V_{k-1}}\delta_{Y^{i,\beta}_s}, \beta^{i_1}_s\right) \\ \vdots \\ \sigma\left(Y^{i_{|V_{k-1}|},\beta}_s, \sum_{i\in V_{k-1}}\delta_{Y^{i,\beta}_s}, \beta^{i_{|V_{k-1}|}}_s\right)\end{pmatrix}, \quad \mathcal{W}^k_s = \begin{pmatrix} W^{i_1}_s \\ \vdots \\ W^{i_{|V_{k-1}|}}_s\end{pmatrix},$$

taking values in $\mathbb{R}^{d|V_{k-1}|}$, $\mathbb{R}^{d|V_{k-1}|}$, $\mathbb{R}^{d|V_{k-1}|\times d'}$, and $\mathbb{R}^{d'|V_{k-1}|}$ respectively. As recalled in Remark 3.2.4, $\mathfrak{b}^k$ and Σ^k are Lipschitz continuous in $\mathbb{R}^{d|V_{k-1}|}$. Therefore, $\mathcal{Y}^k$ is uniquely (up to indistinguishability) defined as the continuous and adapted process satisfying

$$\mathcal{Y}_s^k = \mathcal{Y}_{\tau_{k-1}}^k + \int_{\tau_{k-1}}^s \mathfrak{b}^k(\mathcal{Y}_u^k, \beta_u)du + \int_{\tau_{k-1}}^s \Sigma^k(\mathcal{Y}_u^k, \beta_u)d\mathcal{W}_u^k, \quad \mathbb{P}-\text{a.s.}$$

Describing what happens at branching events τ_k, we can conclude the construction of the branching process. Given the definition of τ_k, there is an (almost surely) unique label, that we denote $\hat{i}_k \in V_{k-1}$, such that

$$Q^{\hat{i}_k}\left((\tau_{k-1}, \tau_k] \times [0, C_\gamma]\right) = 1.$$

Let χ_k the $[0, C_\gamma]$-valued random variable such that (τ_k, χ_k) belongs to the support of $Q^{\hat{i}_k}$. We set V_k as

$$V_k := \begin{cases} V_{k-1}, & \text{if } \chi_k \in \left[\gamma\left(Y_{\tau_k}^{\hat{i}_k,\beta}, \sum_{i\in V_{k-1}} \delta_{Y_{\tau_k}^{i,\beta}}, \beta_{\tau_k}^{\hat{i}_k}\right), C_\gamma\right], \\ V_{k-1}\backslash\left\{\hat{i}_k\right\}, & \text{if } \chi_k \in I_0\left(Y_{\tau_k}^{\hat{i}_k,\beta}, \sum_{i\in V_{k-1}} \delta_{Y_{\tau_k}^{i,\beta}}, \beta_{\tau_k}^{\hat{i}_k}\right), \\ V_{k-1}\backslash\left\{\hat{i}_k\right\} \cup \left\{\hat{i}_k 0, \ldots, \hat{i}_k(\ell-1)\right\}, & \text{if } \chi_k \in I_\ell\left(Y_{\tau_k}^{\hat{i}_k,\beta}, \sum_{i\in V_{k-1}} \delta_{Y_{\tau_k}^{i,\beta}}, \beta_{\tau_k}^{\hat{i}_k}\right) \text{ for } \ell \geq 1, \end{cases}$$

where we impose the continuity of the flow for the off-spring, *i.e.*, $Y_{\tau_k}^{i,\beta} := Y_{\tau_k}^{\hat{i}_k,\beta}$ for $i \in V_k \backslash V_{k-1}$.

We prove that this process satisfies the SDE (3.2.6) by induction. Suppose it holds true up to τ_{k-1}, we have

$$\langle \varphi, \xi_{s\wedge\tau_k}^\beta \rangle = \mathbb{1}_{s\leq\tau_{k-1}} \langle \varphi, \xi_s^\beta \rangle + \mathbb{1}_{\tau_{k-1}<s<\tau_k} \sum_{i\in V_{k-1}} \varphi\left(Y_s^{i,\beta}\right) + \mathbb{1}_{s\geq\tau_k} \sum_{i\in V_k} \varphi\left(Y_{\tau_k}^{i,\beta}\right). \tag{3.2.11}$$

The first term on the r.h.s. satisfies (3.2.6) by induction hypothesis. We apply Itô's formula for each branch to deal with the second one. Finally, the third term is equal to

$$\begin{aligned} \sum_{i\in V_k} \varphi\left(Y_{\tau_k}^{i,\beta}\right) \quad = \quad & \sum_{i\in V_{k-1}} \varphi\left(Y_{\tau_k}^{i,\beta}\right) - \mathbb{1}_{\chi_k \in \left[0,\gamma\left(Y_{\tau_k}^{\hat{i}_k,\beta}, \sum_{i\in V_{k-1}} \delta_{Y_{\tau_k}^{i,\beta}}, \beta_{\tau_k}^{\hat{i}_k}\right)\right)} \varphi\left(Y_{\tau_k}^{\hat{i}_k,\beta}\right) \\ & + \sum_{\ell\geq 1} \mathbb{1}_{\chi_k \in I_\ell\left(Y_{\tau_k}^{\hat{i}_k,\beta}, \sum_{i\in V_{k-1}} \delta_{Y_{\tau_k}^{i,\beta}}, \beta_{\tau_k}^{\hat{i}_k}\right)} \sum_{l=1}^{\ell-1} \varphi\left(Y_{\tau_k}^{\hat{i}_k l,\beta}\right), \end{aligned}$$

which coincides with the integral w.r.t. the Poisson random measures over $(\tau_{k-1}, \tau_k]$. Therefore, (3.2.6) is satisfied up to τ_k and we can conclude by induction.

We focus now on estimates (3.2.7)-(3.2.10). Let $\{\theta_n\}_{n\in\mathbb{N}}$ be defined as follows

$$\theta_n \quad := \quad \inf\{s \geq t : |V_s| \geq n\} \wedge \inf\left\{s \geq t : \sum_{i\in V_u} \left|Y_u^{i,\beta}\right| \geq n\right\}. \tag{3.2.12}$$

The first part of the proof ensures that $\xi_{\cdot\wedge\theta_n}^\beta$ is well-defined and satisfies (3.2.6). Apply (3.2.6) to the function $x \mapsto 1$, obtaining

$$|V_{s\wedge\theta_n}| \quad = \quad |V_t| + \int_{(t,s\wedge\theta_n]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 0} (k-1) \mathbb{1}_{I_k\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right)}(z) Q^i(dudz).$$

Applying Itô's formula, we also obtain

$$\begin{aligned} |V_{s\wedge\theta_n}|^2 &= |V_t|^2 + \int_{(t,s\wedge\theta_n]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 0} \left((|V_u|+k-1)^2 - |V_u|^2\right) \mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z) Q^i(dudz) \\ &= |V_t|^2 + \int_{(t,s\wedge\theta_n]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 0} \left(2(k-1)|V_u| + (k-1)^2\right) \mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z) Q^i(dudz). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \sup_{u\in[t,s]} |V_{u\wedge\theta_n}| &\leq |V_t| + \int_{(t,s\wedge\theta_n]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 1} (k-1) \mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z) Q^i(dudz), \\ \sup_{u\in[t,s]} |V_{u\wedge\theta_n}|^2 &\leq |V_t|^2 + \int_{(t,s\wedge\theta_n]\times\mathbb{R}_+} \sum_{i\in V_{u-}} \sum_{k\geq 1} \left(2(k-1)|V_u| + (k-1)^2\right) \mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z) Q^i(dudz), \end{aligned}$$

and, taking the expectation,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} |V_{u\wedge\theta_n}|\right] &\leq |V_t| + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} \sum_{i\in V_u} \gamma\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right) \sum_{k\geq 1} (k-1) p_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right) du\right] \\ &\leq |V_t| + C_\gamma C_\Phi^1 \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} \sup_{z\in[t,u]} |V_{z\wedge\theta_n}|\right], \\ \mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} |V_{u\wedge\theta_n}|\right] &\leq |V_t| + C_\gamma (C_\Phi^1 + C_\Phi^2) \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} \sup_{z\in[t,u]} |V_{z\wedge\theta_n}|^2\right]. \end{aligned}$$

Applying Grönwall's lemma, we obtain

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} |V_{u\wedge\theta_n}|\right] \leq |V_t| e^{C_\gamma C_\Phi^1 (s-t)}, \qquad \mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} |V_{u\wedge\theta_n}|^2\right] \leq |V_t|^2 e^{C_\gamma (C_\Phi^1 + C_\Phi^2)(s-t)}.$$

Since the bound is uniform in n, θ_n converges almost surely to infinity, and by Fatou's lemma, we retrieve (3.2.7) and (3.2.8). This implies also (3.2.9), since

$$\mathbb{E}^{\mathbb{P}^s}\left[\int_t^s \sum_{i\in V_u} |\beta_u^i| du\right] \leq \mathbb{E}^{\mathbb{P}^s}\left[\int_t^s |V_u| \sup_{i\in\mathcal{I}} |\beta_u^i| du\right] \leq \mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} |V_u| \int_t^s \sup_{i\in\mathcal{I}} |\beta_u^i| du\right] \leq C,$$

where in the last inequality we used Cauchy-Schwartz inequality, (3.2.5) and (3.2.8).

Proving (3.2.10) would be more tricky since the SDE (3.2.6) cannot be applied directly. We see that (3.2.11) is still valid for $\varphi(x) = |x|$. Itô's formula yields, for $s \in (\tau_{k-1}, \tau_k)$,

$$\begin{aligned}
\sum_{i\in V_{k-1}} \left|Y_s^{i,\beta}\right| &= \sum_{i\in V_{k-1}} \left|Y_{\tau_k}^{i,\beta} + \int_{\tau_{k-1}}^{s} b\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right) du + \int_{\tau_{k-1}}^{s} \sigma\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right) dW_u^i\right| \\
&\leq \sum_{i\in V_{k-1}} \left|Y_{\tau_k}^{i,\beta}\right| + \sum_{i\in V_{k-1}} \int_{\tau_{k-1}}^{s} \left|b\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right)\right| du + \sum_{i\in V_{k-1}} \left|\int_{\tau_{k-1}}^{s} \sigma\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right) dW_u^i\right| \\
&\leq \sum_{i\in V_{k-1}} \left|Y_{\tau_k}^{i,\beta}\right| + C_b \int_{\tau_{k-1}}^{s} |V_u| du + C_b \sum_{i\in V_{k-1}} \int_{\tau_{k-1}}^{s} \left(\left|Y_u^{i,\beta}\right| + \left|\beta_u^i\right|\right) du + \\
&\qquad \sum_{i\in V_{k-1}} \left|\int_{\tau_{k-1}}^{s} \sigma\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right) dW_u^i\right|,
\end{aligned}$$

where we have used the bound (3.2.3) over the coefficient b in the last inequality. Since the family of Brownian motions $\{W^i\}_{i\in\mathcal{I}}$ are independent from the one of Poisson measures $\{Q^i\}_{i\in\mathcal{I}}$, we have that taking the conditional expectation with respect to $\mathcal{F}_{\tau_{k-1}}$, we can apply the Burkholder-Davis-Gundy's inequalities (see, *e.g.*, [55, Chapter VII, Theorem 92]). This means that there exists a constant $C > 0$ (which may change from line to line) such that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[\tau_{k-1}\wedge\theta_n, s\wedge\tau_k\wedge\theta_n]} \sum_{i\in V_{k-1}} \left|\int_{\tau_{k-1}\wedge\theta_n}^{u} \sigma\left(Y_r^{i,\beta}, \xi_r^{\beta}, \beta_r^i\right) dW_r^i\right| \middle| \mathcal{F}_{\tau_{k-1}}\right] \\
&\leq C\mathbb{E}^{\mathbb{P}^s}\left[\sum_{i\in V_{k-1}} \left(\int_{\tau_{k-1}\wedge\theta_n}^{s\wedge\tau_k\wedge\theta_n} \mathrm{Tr}\left(\sigma\sigma^\top\left(Y_u^{i,\beta}, \xi_u^{\beta}, \beta_u^i\right)\right) du\right)^{1/2} \middle| \mathcal{F}_{\tau_{k-1}}\right] \\
&\leq C\mathbb{E}^{\mathbb{P}^s}\left[\left(s\wedge\tau_k\wedge\theta_n - \tau_{k-1}\wedge\theta_n\right)|V_{k-1}| \middle| \mathcal{F}_{\tau_{k-1}}\right] = C\mathbb{E}^{\mathbb{P}^s}\left[\int_{\tau_{k-1}\wedge\theta_n}^{s\wedge\tau_k\wedge\theta_n} |V_u| du \middle| \mathcal{F}_{\tau_{k-1}}\right],
\end{aligned}$$

using (3.2.3) in the last line. Therefore, by induction, there exists a constant $C > 0$ (which may change from line to line) such that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} \sum_{i\in V_{u\wedge\theta_n}} \left|Y_{u\wedge\theta_n}^{i,\beta}\right|\right] &\leq \sum_{i\in V} |x^i| + C\Bigg(\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} |V_u| du\right] + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} \sum_{i\in V_u} \left|Y_u^{i,\beta}\right| du\right] \\
&\qquad + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n} \sum_{i\in V_u} \left|\beta_u^i\right| du\right]\Bigg),
\end{aligned}$$

usinig (3.2.7) and (3.2.9) to bound the mass of the population. Applying Grönwall's lemma, we obtain

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]} \sum_{i\in V_{u\wedge\theta_n}} \left|Y_{u\wedge\theta_n}^{i,\beta}\right|\right] \leq C\left(\sum_{i\in V} |x^i| + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^s |V_u| du\right] + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^s \sum_{i\in V_u} \left|\beta_u^i\right| du\right]\right).$$

Since the estimate is uniform in n and θ_n converges almost surely to infinity, applying Fatou's lemma, we retrieve (3.2.10).

□

Control problem

We are given the continuous functions $\psi : \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A \to \mathbb{R}$, $\Psi : M^1(\mathbb{R}^d) \to \mathbb{R}$. We suppose that there exists $C_\Psi, c_\psi > 0$ such that

$$\Psi(\lambda) \le C_\Psi \left(1 + \int_{\mathbb{R}^d} |x|^2 \lambda(dx) + \langle 1, \lambda\rangle^2\right) \tag{3.2.13}$$

$$\Psi(\mu) \ge -C_\Psi \left(1 + \int_{\mathbb{R}^d} |x| \lambda(dx) + \langle 1, \lambda\rangle\right) \tag{3.2.14}$$

$$\psi(x, \lambda, a) \le C_\Psi \left(1 + |x|^2 + \int_{\mathbb{R}^d} |x| \lambda(dx) + |a|^2\right) \tag{3.2.15}$$

$$\psi(x, \lambda, a) \ge -C_\Psi (1 + |x|) + c_\psi |a|^2 \tag{3.2.16}$$

for $\lambda \in M^1(\mathbb{R}^d)$.

Fix a standard strong control β and $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ a starting condition. We define the cost function as

$$J(t, \lambda; \beta) := \mathbb{E}^{\mathbb{P}^s}\left[\int_t^T \sum_{i \in V_s} \psi\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) ds + \Psi\left(\xi_T^\beta\right) \middle| \xi_t^\beta = \lambda\right].$$

As the dependence of the cost J on the label is solely through the spatial components and the control, we limit the set of controls. This restriction is implemented to maintain symmetry between positions in $\mathbb{R}^d$ and the chosen control in A, enabling a natural embedding of strong controls into relaxed ones.

Definition 3.2.6 (Admissible strong control)**.** *Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. We say that $\beta = (\beta^i)_{i \in \mathcal{I}}$ is an* admissible strong control*, and we denote $\beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$, if β is a standard strong control and*

$$\mathbb{E}^{\mathbb{P}^s}\left[\int_t^T \sum_{i,j \in V_s, i \neq j} \mathbb{1}_{Y_s^{i,\beta} = Y_s^{j,\beta}, \beta_s^i \neq \beta_s^j} ds\right] = 0. \tag{3.2.17}$$

We can now state the *strong control problem* as

$$v^s(t, \lambda) = \inf\left\{J(t, \lambda; \beta) : \beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}\right\}, \tag{3.2.18}$$

for $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$.

Remark 3.2.5. *Under additional assumptions, restricting from standard to admissible controls does not impact the value function. For example, whenever σ is uniformly elliptic, i.e., there exist $\varepsilon > 0$ such that $\sigma\sigma^\top(x, \lambda, a) \ge \varepsilon \mathbb{I}_d$, with $\mathbb{I}_d$ being the identity matrix of dimension $d \times d$, all alive particles take different positions $dt \otimes d\mathbb{P}$-a.s. Therefore, all standard controls are admissible.*

3.2.3 Well-posedness of the control problem

To finally give a well-posedness of the control problem, we must prove the finite second order of the Branching Processes, at least close to an optimal value. We apply the techniques used to prove Proposition 3.2.13 to get the next lemma.

Lemma 3.2.4. *Let $(t,\lambda)\in\mathbb{R}_+\times\mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. There exists a constant $C>0$ depending only on T and on the coefficients b, σ, γ and $(p_k)_k$ such that*

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,t+h]}\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2\right] \leq C\Bigg(\sum_{i\in V}|x^i|^2+\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}|V_u|du\right] + \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}\sum_{i\in V_u}\left|\beta_u^i\right|^2du\right]\Bigg), \tag{3.2.19}$$

for any $h>0$.

Proof. Fix $\left(t,\lambda=\sum_{i\in V}\delta_{x^i}\right)\in\mathbb{R}_+\times\mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be as in (3.2.12). We have that $\xi^\beta_{\cdot\wedge\theta_n}$ satisfies (3.2.6). Applying (3.2.6) to the function $x\mapsto|x|^2$, we get

$$\begin{aligned}\sum_{i\in V_{s\wedge\theta_n}}\left|Y_{s\wedge\theta_n}^{i,\beta}\right|^2 &= \sum_{i\in V}|x^i|^2+\int_t^{s\wedge\theta_n}\sum_{i\in V_u}2\left(Y_u^{i,\beta}\right)^\top\sigma\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)dB_u^i\\ &+\int_t^{s\wedge\theta_n}\sum_{i\in V_u}2\left(Y_u^{i,\beta}\right)^\top b\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du+\\ &+\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\mathrm{Tr}\left(\sigma\sigma^\top\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)\right)du\\ &+\int_{(t,s\wedge\theta_n]\times\mathbb{R}_+}\sum_{i\in V_{u-}}\sum_{k\geq0}(k-1)\left|Y_u^{i,\beta}\right|^2\mathbb{1}_{I_k\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)}(z)Q^i(dudz)\ ,\end{aligned}$$

Taking the supremum in the interval $[t,s]$ and taking the expectation, we bound each term in the r.h.s. Applying Burkholder-Davis-Gundy's inequalities to the second term, there exists a constant $C>0$ (which may change from line to line) such that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]}\int_t^{u\wedge\theta_n}\sum_{i\in V_r}2\left(Y_r^{i,\beta}\right)^\top\sigma\left(Y_r^{i,\beta},\xi_r^\beta,\beta_r^i\right)dB_r^i\right]\\ &\leq C\mathbb{E}^{\mathbb{P}^s}\left[\left(\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2\mathrm{Tr}\left(\sigma\sigma^\top\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)\right)du\right)^{1/2}\right]\leq C\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2du\right].\end{aligned}$$

From (3.2.3) on the growth of b and σ, the third and the fourth terms can be bounded as follows

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]}\int_t^{u\wedge\theta_n}\sum_{i\in V_r}\left(2\left(Y_r^{i,\beta}\right)^\top b\left(Y_r^{i,\beta},\xi_r^\beta,\beta_r^i\right)+\mathrm{Tr}\left(\sigma\sigma^\top\left(Y_r^{i,\beta},\xi_r^\beta,\beta_r^i\right)\right)\right)dr\right]\\ &\qquad\leq C\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n}|V_u|+\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2+\left|\beta_u^i\right|^2du\right],\end{aligned}$$

using that $a^\top b \le \frac{1}{2}\left(|a|^2+|b|^2\right)$ for $a,b\in\mathbb{R}^d$. Finally, the last term gives

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,s]}\int_{(t,u\wedge\theta_n]\times\mathbb{R}_+}\sum_{i\in V_{r-}}\sum_{k\ge 0}(k-1)\left|Y_r^{i,\beta}\right|^2\mathbb{1}_{I_k\left(Y_r^{i,\beta},\xi_r^\beta,\beta_r^i\right)}(z)Q^i(drdz)\right]$$
$$\le \mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n}\sum_{i\in V_{u-}}\gamma\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)\sum_{k\ge1}(k-1)\left|Y_u^{i,\beta}\right|^2 p_k\left(Y_u^{i,\beta},\xi_u^\beta\beta_u^i\right)du\right]$$
$$\le C\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2du\right].$$

Combining all the terms and using Grönwall's inequality first and Fatou's lemma, we obtain (3.2.19). □

This lemma tells us that whenever $\mathbb{E}^{\mathbb{P}^s}\left[\int_t^T\sum_{i\in V_u}\left|\beta_u^i\right|^2du\right]<\infty$, we have $|J(t,\lambda;\beta)|<\infty$ from the coercivity bounds. Therefore, ε-optimal controls must satisfy this condition, as shown in the following proposition.

Proposition 3.2.14. *Fix $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$. Let $\varepsilon>0$, and let $\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\lambda)}$ be the set of $\beta\in\mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$ satisfying*

$$J(t,\lambda;\beta)\le v^s(t,\lambda)+\varepsilon.$$

Then

$$\sup_{\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\lambda)}}\mathbb{E}^{\mathbb{P}^s}\left[\int_t^T\sum_{i\in V_u}\left|\beta_u^i\right|^2du\right]<\infty. \tag{3.2.20}$$

Moreover, $v^s(t,\lambda)>-\infty$.

Proof. We use (3.2.14) and (3.2.16) along with Lemma 3.2.4 to find a constant $C>0$ (which may change from line to line) such that, for all $\beta\in\mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$,

$$\begin{aligned}J(t,\lambda;\beta) &\ge -C\mathbb{E}^{\mathbb{P}^s}\left[1+\sup_{u\in[t,T]}|V_u|^2+\sup_{u\in[t,T]}\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|\right]+c_\psi\mathbb{E}^{\mathbb{P}^s}\left[\int_t^T\sum_{i\in V_u}\left|\beta_u^i\right|^2du\right]\\ &\ge -C\mathbb{E}^{\mathbb{P}^s}\left[1+\int_t^T\sum_{i\in V_u}\left|\beta_u^i\right|du\right]+c_\psi\mathbb{E}^{\mathbb{P}^s}\left[\int_t^T\sum_{i\in V_u}\left|\beta_u^i\right|^2du\right]\end{aligned} \tag{3.2.21}$$

This already proves $v^s(t,\lambda)>-\infty$, as the function $a\mapsto c_\psi|a|^2-C|a|$ is bounded from above. To prove the first claim, fix arbitrarily a constant control $\beta_s^{a_0,i}:=a_0\in A$. Lemma 3.2.4 and Proposition 3.2.13 imply

$$\mathbb{E}^{\mathbb{P}^s}\left[\sup_{u\in[t,t+h]}\sum_{i\in V_u}\left|Y_u^{i,\beta^{a_0}}\right|^2\right] \le C\left(1+\mathbb{E}^{\mathbb{P}^s}\left[\int_t^{t+h}\sum_{i\in V_u}\left|\beta_u^{a_0,i}\right|^2du\right]\right)\le C\left(1+|a_0|^2\right).$$

Then, from (3.2.13) and (3.2.15), we have show $J(t,\lambda;\beta^{a_0})<\infty$. Therefore, for $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\lambda)}$, we have $J(t,\lambda;\beta) \le J(t,\lambda;\beta^{a_0}) + \varepsilon$. This and (3.2.21) yield

$$\sup_{\beta\in\mathcal{R}^{s,\varepsilon}_{(t,\lambda)}} \mathbb{E}^{\mathbb{P}^s}\left[\int_t^T \sum_{i\in V_u}\left(\left|\beta_u^i\right|^2 - C\left|\beta_u^i\right|\right)du\right] < \infty.$$

This gives (3.2.20), by Proposition 3.2.13. □

3.3 Relaxed formulation

We give the relaxed formulation for the branching diffusion control problem by working with relaxed controls and weak solutions of the previous SDE.

We equip the product space $[0,T]\times\mathbb{R}^d\times A$ with the σ-algebra $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(A)$. Let $\mathcal{A}^{\mathrm{Leb}} \subseteq M^1([0,T]\times\mathbb{R}^d\times A)$ be the set of measures, whose projection on $[0,T]$ is the Lebesgue measure. Each $\alpha\in\mathcal{A}^{\mathrm{Leb}}$ can be identified with its disintegration (see, *e.g.*, [101, Corollary 1.26, Chapter 1]). In particular, we have $\alpha(ds,dx,da) = ds\mathbf{y}_s(dx)\bar{\alpha}_s(x,da)$, for a process $(\mathbf{y}_s(dx))_s$ (resp. $(\bar{\alpha}_s(x,da))_s$) taking values in the set of functions from $[0,T]$ (resp. $[0,T]\times\mathbb{R}^d$) to $M^1(\mathbb{R}^d)$ (resp. $M^1(A)$). Let $\mathcal{A}^{\mathrm{Leb},\cdot,1}\subseteq\mathcal{A}^{\mathrm{Leb},\cdot,\cdot}$ be the set of elements α such that $\bar{\alpha}_s(x,da)\in\mathcal{P}^1(A)$ for any $(s,x)\in[0,T]\times\mathbb{R}^d$. For $\mathbf{x}=(\mathbf{x}_s)_s\in\mathbf{D}^d$ fixed, we denote the *space of relaxed controls* $\mathcal{A}^{\mathrm{Leb},\mathbf{x},1}$ as

$$\mathcal{A}^{\mathrm{Leb},\mathbf{x},1} := \left\{\alpha\in\mathcal{A}^{\mathrm{Leb},\cdot,1} : \alpha(ds,dx,da) = ds\mathbf{x}_s(dx)\bar{\alpha}_s(x,da) \text{ a.e. } s\in[0,T]\right\},$$

which is *weakly* closed.*

3.3.1 Martingale model

Let $\mathcal{L}$ be the generator defined on the cylindrical functions $F_\varphi = F(\langle\varphi,\cdot\rangle)$, for $F\in C_b^2(\mathbb{R})$ and $\varphi\in C_b^2(\mathbb{R}^d)$, as

$$\begin{aligned}\mathcal{L}F_\varphi(x,\lambda,a) &= F'(\langle\varphi,\lambda\rangle)L\varphi(x,\lambda,a) + \frac{1}{2}F''(\langle\varphi,\lambda\rangle)\left|D\varphi(x)\sigma(x,\lambda,a)\right|^2\\ &\quad+\gamma(x,\lambda,a)\left(\sum_{k\ge 0}F\big(\langle\varphi,\lambda\rangle+(k-1)\varphi(x)\big)p_k(x,\lambda,a) - F_\varphi(\lambda)\right).\end{aligned}$$

For simplicity, we write $F'_\varphi(\lambda)$ for $F'(\langle\varphi,\lambda\rangle)$ and $F''_\varphi(\lambda)$ for $F''(\langle\varphi,\lambda\rangle)$. Moreover, for $\mathbb{F}=\{\mathcal{F}_s\}_{s\ge 0}$ a filtration, we denote $\hat{\mathbb{F}}=\left\{\hat{\mathcal{F}}_s\right\}_{s\ge0}$ the filtration such that $\hat{\mathcal{F}}_s := \mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}_s$ for any $s\ge 0$.

Definition 3.3.7 (Relaxed control)**.** *Fix $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$. We say that $\mathcal{C}$ is a* relaxed control*, and we denote $\mathcal{C}\in\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$, if*

$$\mathcal{C} = \left(\Omega,\mathcal{F},\mathbb{P},\mathbb{F}=\{\mathcal{F}_s\}_{s\ge0},(X_s)_{s\ge0},(\bar{\alpha}_s)_{s\ge0}\right)$$

where

(i) $(\Omega,\mathcal{F},\mathbb{P})$ *is a probability space with filtration* $\mathbb{F}$*;*

(ii) $(X_s)_{s\ge0}$ *is an* $\mathbb{F}$*-progressively measurable process living in* $\boldsymbol{D}^d$ *such that* $\mathbb{P}(X_t=\lambda)=1$*;*

(iii) $\bar{\alpha} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathcal{P}^1(A)$ *is a* $\hat{\mathbb{F}}$*-predictable process associated with* $\alpha \in \mathcal{A}^{Leb,\cdot,1}$ *such that* $\mathbb{P}(\alpha \in \mathcal{A}^{Leb,X,1}) = 1$, *i.e.,*

$$\mathbb{P}\big(\alpha(ds,dx,da) = dsX_s(dx)\bar{\alpha}_s(x,da) \ a.e. \ s \in [0,T]\big) = 1,$$
$$\mathbb{E}^{\mathbb{P}}\left[\int_t^T \int_{\mathbb{R}^d \times A} |a|\bar{\alpha}_s(x,da)X_s(dx)ds\right] < \infty;$$

(iv) for any $F_\varphi = F(\langle \varphi, \cdot \rangle)$, *with* $F \in C_b^2(\mathbb{R})$ *and* $\varphi \in C_b^2(\mathbb{R}^d)$, *the process*

$$M_s^{F_\varphi} = F_\varphi(X_s) - \int_t^s \int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(x, X_u, a)\bar{\alpha}_u(x,da)X_u(dx)du \tag{3.3.22}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale for* $s \geq t$.

Remark 3.3.6. *We highlight two main aspects of this definition.*

1. *For* $\mathcal{C} \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$, *we are only interested in the time interval* $[t,T]$. *Therefore,* X_s *and* α_s *can be redefined for* $s \in [0,t)$ *as* $X_s = \lambda$ *and* $\alpha_s = \delta_{a_0}$ *for some* $a_0 \in A$.

2. *For* $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$, *admissible strong controls are embedded in* $\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$. *Indeed, it suffices to consider* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ *as in Section 3.2 and define* $(\bar{\alpha}_s)_s$ *as* $\bar{\alpha}_s(x,da) = \delta_{\mathfrak{a}(s,x)}$ *for*

$$\mathfrak{a}(s,x) := \frac{\sum_{i \in V_{s-}} \beta^i_{s-} \mathbb{1}_{Y^{i,\beta}_{s-} = x}}{\sum_{i \in V_{s-}} \mathbb{1}_{Y^{i,\beta}_{s-} = x}} \mathbb{1}_{\{|V_{s-}|>0\}} + a_0 \mathbb{1}_{\{|V_{s-}|=0\} \cup \{s \leq t\}}, \tag{3.3.23}$$

for some $a_0 \in A$ *and with the convention* $0/0 := a_0$. *The SDE* (3.2.6), *combined with Itô's formula for semimartingales, implies* (3.3.22). *Hence, it is a relaxed control, and, with abuse of notation we denote* $\beta \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$.

We can find equivalent representations of (3.3.22), an important tool in the manipulation of these objects. It is given using the quadratic variation of a martingale (see, *e.g.*, [98, Chapter I-4e]).

Lemma 3.3.5. *Given* $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$, *let* $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s, (X_s)_s, (\alpha_s)_s)$ *be such that conditions (i), (ii), and (iii) in the definition 3.3.7 are satisfied. The following are equivalent.*

(i) We have $\mathcal{C} \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$.

(ii) For any $\varphi \in C_b^2(\mathbb{R}^d)$ *such that* $\varphi > \varepsilon$ *for some* $\varepsilon > 0$ *and* $\sup_{\mathbb{R}^d} \varphi \leq 1$,

$$M_s^{\exp_{\log\varphi}} = e^{\langle \log\varphi, X_s\rangle} - \int_t^s \int_{\mathbb{R}^d \times A} \left(\frac{L\varphi(x,X_u,a) + \gamma(x,X_u,a)(\Phi(\varphi(x),x,X_u,a) - \varphi(x))}{\varphi(x)} \right) \bar{\alpha}_u(x,da)X_u(dx)\, e^{\langle \log\varphi, X_u\rangle} du \tag{3.3.24}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale for* $s \geq t$.

(iii) For any $\varphi \in C_b^2(\mathbb{R}^d)$ *the process*

$$\begin{aligned} \bar{M}_s^{\varphi} = \langle \varphi, X_t \rangle \quad & - \quad \int_t^s \int_{\mathbb{R}^d \times A} L\varphi(x, X_u, a)\bar{\alpha}_u(x, da)X_u(dx)du \\ & - \quad \int_t^s \int_{\mathbb{R}^d \times A} \gamma(x, X_u, a)\left(\partial_s \Phi(1, x, X_u, a) - 1\right)\varphi(x) \\ & \qquad \bar{\alpha}_u(x, da)X_u(dx)du \ , \quad s \in [t, T]. \end{aligned} \tag{3.3.25}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale with quadratic variation process*

$$\begin{aligned} \left[\bar{M}^{\varphi}\right]_s \quad = \quad & \int_t^s \int_{\mathbb{R}^d \times A} \Big(Tr\left(\sigma\sigma^\top(x, X_u, a) D\varphi D\varphi^\top(x)\right) \\ & + \gamma(x, X_u, a)\left(\partial_{ss}^2 \Phi(1, x, X_u, a) - \partial_s \Phi(1, x, X_u, a) + 1\right)\varphi^2(x)\Big) \\ & \qquad \bar{\alpha}_u(x, da)X_u(dx)du \ , \quad s \in [t, T]. \end{aligned} \tag{3.3.26}$$

Proof. $(i) \implies (ii)$: We need to prove that (3.3.22) is a martingale for the function $F_{\log \varphi}$ with $F(x) = \exp(x)$ and $\varphi \in C_b^2(\mathbb{R}^d)$ such that $\varphi > \varepsilon$ for some $\varepsilon > 0$ and $\sup_{\mathbb{R}^d} \varphi \le 1$. The process $M^{\exp_{\log \varphi}}$, as in (3.3.24), is a local martingale. To prove that it is a martingale, we show its quadratic variation has a finite expectation. Since the compensator of $(M^{\exp_{\log \varphi}})^2$ is the same of $M^{\exp_{2\log\varphi}} = M^{\exp_{\log \varphi^2}}$, we get the quadratic variation of $M^{\exp_{\log \varphi}}$ applying (3.3.22) to $F \in C_b^2(\mathbb{R})$ and φ^2. Therefore, it is equal to

$$\begin{aligned} [M^{\exp_{\log \varphi}}]_s = \int_t^s \int_{\mathbb{R}^d \times A} \left(\frac{L\varphi^2(x, X_u, a) + \gamma(x, X_u, a)(\Phi(\varphi^2(x), x, X_u, a) - \varphi^2(x))}{\varphi^2(x)} \right) \\ \bar{\alpha}_u(x, da)X_u(dx)\ e^{\langle \log \varphi^2, X_u \rangle} du. \end{aligned}$$

Since $[M^{\exp_{\log \varphi}}]$ is uniformly bounded, using Itô's isometry, $M^{\exp_{\log \varphi}}$ is a martingale.

$(ii) \implies (iii)$: Fix $f \in C_b^2(\mathbb{R}^d)$. For $\theta > 0$, and $M_f := \sup_{\mathbb{R}^d} |f|$, we define $\varphi_1 := e^{\theta(f - M_f)}$ and $\varphi_2 := e^{-\theta M_f}$. Since f is bounded, there exists $\varepsilon > 0$ such that $\varphi_1 > \varepsilon$ and $\sup_{\mathbb{R}^d} \varphi_1 \le 1$. Applying (3.3.24) to φ_1 and φ_2, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\Bigg[& e^{\langle \theta(f - M_f), X_{s+h} \rangle} - e^{\langle \theta(f - M_f), X_s \rangle} \\ & - \int_s^{s+h} \int_{\mathbb{R}^d \times A} \Big(\theta Lf(x, X_u, a) + \theta^2 \mathrm{Tr}\left(\sigma\sigma^\top(x, X_u, a) Df Df^\top(x)\right) \\ & \qquad + \gamma(x, X_u, a) \frac{\Phi\left(\left(e^{\theta(f(x) - M_f)}\right), x, X_u, a\right) - e^{\theta(f(x) - M_f)}}{e^{\theta(f(x) - M_f)}} \Big) \\ & \qquad \bar{\alpha}_u(x, da)X_u(dx) e^{\langle \theta(f - M_f), X_u \rangle} du \Bigg| \mathcal{F}_s \Bigg] = 0 \ , \end{aligned} \tag{3.3.27}$$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\Bigg[& e^{\langle -\theta M_f, X_{s+h} \rangle} - e^{\langle -\theta M_f, X_s \rangle} - \int_s^{s+h} \int_{\mathbb{R}^d \times A} \gamma(x, X_u, a) \\ & \frac{\Phi\left(e^{-\theta M_f}, x, X_u, a\right) - \left(e^{-\theta M_f}\right)}{e^{-\theta M_f}} \bar{\alpha}_u(x, da)X_u(dx) e^{\langle -\theta M_f, X_u \rangle} du \Bigg| \mathcal{F}_s \Bigg] = 0 \ . \end{aligned} \tag{3.3.28}$$

Since all the functions are bounded, we are allowed to differentiate with respect to θ. Dividing by θ, subtracting (3.3.27) and (3.3.28), and setting $\theta = 0$, we get (3.3.25). Differentiating twice with respect to θ, dividing by θ^2subtracting (3.3.27) and (3.3.28) and setting $\theta = 0$, we get (3.3.26).

$(iii) \implies (i)$: We prove the last implication using Itô's formula for semimartingales. Fix $F \in C^2(\mathbb{R}^n)$ and $f \in C_b^2(\mathbb{R}^n)$. We have that $\langle f, X_s \rangle_{s \geq t}$ is a $\mathbb{P}$-semimartingale, and so, by Itô's formula, we have (3.3.22). □

3.3.2 Representation and relaxed control problem

In this section, we show that relaxed controls can be expressed as solutions to stochastic differential equations. This representation proves valuable in establishing the non-explosion property and, subsequently, the well-posedness of the control problem. This characterization relies on martingale measures within extensions of the designated space. Succinct definitions and pertinent results concerning these entities are summarized in [67] (for a comprehensive study on the subject, refer to [153]). Here, we provide a brief recap of their definition.

Definition 3.3.8. *Let $(G, \mathcal{G})$ be a Lusin space with its σ-algebra, and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s)$ a filtered space satisfying the usual condition, where we define $\mathcal{P}$ the predictable σ-field. A process $\mathcal{M}$ on $\Omega \times [0,T] \times \mathcal{G}$ is called* martingale measure on G *if*

(i) $\mathcal{M}_0(E) = 0$ a.s. for any $E \in \mathcal{G}$;

(ii) $\mathcal{M}_t$ is a σ-finite, $L^2(\Omega)$-valued measure for all $t \in [0,T]$;

(iii) $(\mathcal{M}_t(E))_{t\in[0,T]}$ is an $\mathbb{F}$-martingale for any $E \in \mathcal{G}$.

We say that $\mathcal{M}$ is orthogonal *if the product $\mathcal{M}_t(E)\mathcal{M}_t(E')$ is a martingale for any two disjoint sets $E, E' \in \mathcal{G}$. We also say, on one hand, that is* continuous *if $(\mathcal{M}_t(E))_{t\geq 0}$ is continuous,* purely discontinuous*, on the other hand, if $(\mathcal{M}_t(E))_{t\geq 0}$ is a purely discontinuous martingale for any $E \in \mathcal{G}$.*

For a strong representation of relaxed controls, we rely on the notion of *predictable projection* and *intensity* that we briefly recall. For an $\mathbb{R}$-valued $\mathbb{F}$-adapted process Y, there exists (see, *e.g.*, [98, Theorem 2.28, Chapter I]) a $(-\infty, \infty]$-valued process, called the *predictable projection* of Y and denoted by ${}^P Y$. It is determined uniquely up to a negligible set by the following two conditions:

(i) it is predictable;

(ii) ${}^P Y_T = \mathbb{E}^{\mathbb{P}}[Y_T | \mathcal{F}_{T-}]$ on $\{T < \infty\}$ for all predictable stopping times T.

For a continuous orthogonal martingale measure $\mathcal{M}$ on G, there exists a random, predictable real-valued measure I on $\mathcal{B}([0,T]) \otimes \mathcal{G}$, called *intensity* of $\mathcal{M}$, defined by: $[\mathcal{M}(E)]_s = \int_0^t \int_E I(dx, ds)$ $\mathbb{P}$-a.s., for all $t > 0$. We can construct a stochastic integral with respect to $\mathcal{M}$ for all functions φ defined on $\Omega \times [0,T] \times G$, $\mathcal{P} \otimes \mathcal{G}$ measurable, such that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_E \varphi^2(\omega, s, x) I(\omega, dx, ds)\right] < \infty,$$

denoted by $\int_0^t \int_E \varphi(s,x)\mathcal{M}(dx, ds)$. We refer to [153, Chapter 2] for the proofs.

The representation of these processes is grounded in the representation theorems for continuous and purely discontinuous martingale measures, as done in [121]. We apply her construction in our context and get the following proposition.

Proposition 3.3.15. *Let* $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s, (X_s)_s, (\alpha_s)_s) \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$. *There exists an extension* $\left(\hat{\Omega} = \Omega \times \tilde{\Omega}, \hat{\mathcal{F}} = \mathcal{F} \otimes \tilde{\mathcal{F}}, \hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}, \left\{\hat{\mathcal{F}}_s = \mathcal{F}_s \otimes \tilde{\mathcal{F}}_s\right\}_s\right)$ *of* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, *where we naturally extend* X *and* α, *that satisfies the following properties.*

1. $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ *is a filtered probability space supporting a continuous* $\hat{\mathbb{F}}$*-martingale measures* $\mathcal{M}^c$ *on* $\hat{\Omega} \times [0,T] \times \mathbb{R}^d \times A$, *with intensity measure* $dsX_s(dx)\bar{\alpha}_s(x, da)$, *and a purely discontinuous* $\hat{\mathbb{F}}$*-martingale measure* $\mathcal{M}^d$ *on* $\hat{\Omega} \times [0,T] \times \mathbb{R}^d \times \mathbb{R}_+ \times A$, *with dual predictable projection measure* $dsX_s(dx)dz\bar{\alpha}_s(x, da)$.

2. $\hat{\mathbb{P}} \circ X_t^{-1} = \lambda$.

3. $\hat{\mathbb{P}}(\alpha \in \mathcal{A}^{Leb,X,1}) = 1$.

4. X *satisfies the following dynamics*

$$\begin{aligned}
\langle f, X_s\rangle = \langle f, \lambda\rangle \quad &+ \int_t^s \int_{\mathbb{R}^d\times A} \big(Lf(x, X_r, a) + \\
&\qquad \gamma(x, X_r, a)\left(\partial_s\Phi(1, x, X_r, a) - 1\right) f(x)\big)\bar{\alpha}_r(x, da)X_r(dx)dr \\
&+ \int_t^s \int_{\mathbb{R}^d\times A} Df(x)\sigma(x, X_s, a)\mathcal{M}^c(dr, dx, da) \qquad (3.3.29)\\
&+ \int_t^s \int_{\mathbb{R}^d\times\mathbb{R}_+\times A} \sum_{k\geq 0}\langle f, (k-1)\delta_x\rangle \mathbb{1}_{I_k(x,X_r,a)}(z)\mathcal{M}^d(dr, dx, dz, da)\ .
\end{aligned}$$

for all $f \in C_b^\infty(\mathbb{R}^d)$ *and all* $[t, s] \subseteq [0, T]$.

Proof. We follow the ideas in [121, Theorem 2.7] and [121, Theorem 2.9] to characterize the martingale $\bar{M}_s^f$ in (3.3.25). From [98, Theorem 4.18], every square integrable martingale starting at 0 can be uniquely decomposed in the sum of a continuous martingale $\bar{M}^{f,c}$ and a purely discontinuous martingale $\bar{M}^{f,d}$, which is the compensated sum of its jumps. We show the connection of these two processes with X and α.

First, we focus on $\bar{M}^{f,d}$. Since a purely discontinuous martingale $\bar{M}^{f,d}$ is the compensated sum of its jumps, we look at $\Delta X_s = X_s - X_{s-}$. Let $\tilde{N}$ be the Lévy system of X, *i.e.*, a measure on $M^1(\mathbb{R}^d) \times \mathbb{R}_+$ given by $N_s(X_s, dv)ds$ where $N_s(\bar{X}, dv)$ is the image measure of the measure $\nu_s(x, \bar{X}, du)\bar{X}(dx)$ by the mapping $(u, x) \mapsto u\delta_x$ from $\mathbb{R}_+ \times \mathbb{R}^d$ to $M^1(\mathbb{R}^d)$, and a certain kernel ν. Comparing the last term in expressions (3.3.22) and [68, Théorème 7 (4)], we identify ν as

$$\nu_s(x, \lambda, dz) = \int_A \sum_{k\geq 0} (k-1)\, \mathbb{1}_{I_k(x,\lambda,a)}(z)\bar{\alpha}_s(x, da)dz.$$

This means that, for F bounded positive measurable function on $\mathbb{R}_+ \times M^1(\mathbb{R}^d)$, we have that

$$\begin{aligned}
&\sum_{t<r\leq s} F(r,\Delta X_r)\mathbb{1}_{\{\Delta X_r\neq 0\}} \\
&\qquad - \int_t^s \int_{\mathbb{R}^d}\int_{(0,\infty)}\int_A \sum_{k\geq 0} F(r,(k-1)\delta_x)\mathbb{1}_{I_k(x,X_r,a)}(z)\bar{\alpha}_r(x,da)dzX_r(dx)dr \\
&= \sum_{t<r\leq s} F(r,\Delta X_r)\mathbb{1}_{\{\Delta X_r\neq 0\}} \qquad (3.3.30)\\
&\qquad - \int_t^s \int_{\mathbb{R}^d\times A} \sum_{k\geq 0} F(r,(k-1)\delta_x)\gamma(x,X_r,a)p_k(x,X_r,a)\bar{\alpha}_r(x,da)X_r(dx)dr
\end{aligned}$$

is a $\mathbb{F}$-martingale. With this description of ν and $N_s(X_s,dv)ds$, we use [121, Proposition 2.8] to prove that we satisfy the hypothesis of [66, Theorem 12]. Therefore, there exists an extension $\left(\bar{\Omega}^1 = \Omega\times\Omega^1, \bar{\mathcal{F}}^1 = \mathcal{F}\otimes\mathcal{F}^1, \bar{\mathbb{P}}^1 = \mathbb{P}\otimes\mathbb{P}^1, \left\{\bar{\mathcal{F}}^1_s = \mathcal{F}_s\otimes\mathcal{F}^1_s\right\}_s\right)$, and martingale measures $\mathcal{M}^d$ on $[0,T]\times\mathbb{R}^d\times\mathbb{R}_+\times A$ in it, such that its dual predictable projection measure is $drX_r(dx)dz\bar{\alpha}_r(x,da)$, and

$$\bar{M}^{f,d}_s = \int_t^s \int_{\mathbb{R}^d\times\mathbb{R}_+\times A} \sum_{k\geq 0}\langle f,(k-1)\delta_x\rangle \mathbb{1}_{I_k(x,X_r,a)}(z)\mathcal{M}^d(dr,dx,dz,da).$$

Focus now on $\bar{M}^{f,c}$. The first term in (3.3.26) comes from the continuous martingale, *i.e.*,

$$\left[\bar{M}^{f,c}\right]_s = \int_t^s \int_{\mathbb{R}^d\times A} \operatorname{Tr}\left(\sigma\sigma^\top(x,X_r,a)D\varphi D\varphi^\top(x)\right)\bar{\alpha}_r(x,da)X_r(dx)dr.$$

Since $\sigma\in L^2(X_s(dx)\alpha_s(da)ds)$, from [67, Theorem III-7], there exist an extension $\left(\bar{\Omega}^2 = \bar{\Omega}^1\times\Omega^2, \bar{\mathcal{F}}^2 = \bar{\mathcal{F}}^1\otimes\mathcal{F}^2, \bar{\mathbb{P}}^2 = \bar{\mathbb{P}}^1\otimes\mathbb{P}^2, \left\{\bar{\mathcal{F}}^2_s = \bar{\mathcal{F}}^1_s\otimes\mathcal{F}^2_s\right\}_s\right)$, and a continuous martingale measure $\mathcal{M}^c$ on $[0,T]\times\mathbb{R}^d\times A$ on this space, such that its intensity is $dsX_s(dx)\bar{\alpha}_s(x,da)$, and we have

$$\bar{M}^{f,c}_s = \int_t^s \int_{\mathbb{R}^d\times A} Df(x)\sigma(x,X_r,a)\mathcal{M}^c(dr,dx,da).$$

The imposed dependence on X and α over $\mathcal{M}^d$ and $\mathcal{M}^c$ implies that (3.3.29) is satisfied.

Conversely, if a $M^1(\mathbb{R}^d)$-valued process satisfies (3.3.29), applying Itô's formula, we have (3.3.24). □

We can now define the relaxed control problem. For $\mathcal{C}\in\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$, we define the cost function as

$$J(t,\lambda;\mathcal{C}) = \mathbb{E}^{\mathbb{P}}\left[\int_t^T \int_{\mathbb{R}^d\times A} \psi\left(s,X_s,a\right)\bar{\alpha}_s(x,da)X_s(dx)ds + \Psi\left(X_T\right)\right], \qquad (3.3.31)$$

and the *relaxed control problem* as

$$v^r(t,\lambda) = \inf\left\{J(t,\lambda;\mathcal{C}) : \mathcal{C}\in\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}\right\}, \qquad (3.3.32)$$

for any $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$.

To achieve the well-posedness of this problem, akin to the case of strong controls, it is necessary to get non-explosion bounds, as presented in Proposition 3.2.13 and Proposition 3.2.14. However, we choose an alternative approach instead of replicating similar results within this new framework. Firstly, we establish an equivalence between the strong and relaxed formulations. Subsequently, we employ this equivalence to retrieve estimates for the relaxed formulation, thereby ensuring the well-posedness of the relaxed control problem.

3.4 Equivalence between strong and relaxed formulation

We state the following straightforward adaptation of [86, Lemma 3.7]. This enables the process X to be reduced to its canonical filtration. It is important to emphasize that the following lemma is presented in relation to the filtration generated by the processes, rather than its right-continuous extension or its completion with respect to a specific probability measure. This construction aligns with the approach described in [86], where the only requirement is the existence of a countably dense set of test functions that define the martingale problem.

Lemma 3.4.6. *Fix $(t,\lambda) \in [0,T]\times\mathcal{N}[\mathbb{R}^d]$ and $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq 0}, (X_s)_{s\geq 0}, (\bar{\alpha}_s)_{s\geq 0}) \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$. If $\left\{\mathcal{F}^X_s\right\}_s$ is the filtration generated by X and $\{\mathcal{G}_s\}_{s\geq 0}$ another filtration such that $\mathcal{F}^X_s \subseteq \mathcal{G}_s \subseteq \mathcal{F}_s$ for any $s \geq 0$. Then, there exists $\left(\bar{\alpha}^{\mathcal{G}}_s\right)_{s\geq 0}$ such that*

$$\bar{\mathcal{C}} = \left(\Omega, \mathcal{G}_T, \mathbb{P}, \{\mathcal{G}_s\}_{s\geq 0}, (X_s)_{s\geq 0}, \left(\bar{\alpha}^{\mathcal{G}}_s\right)_{s\geq 0}\right)$$

is in $\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$ and $J(t,\lambda;\mathcal{C}) = J(t,\lambda;\bar{\mathcal{C}})$.

Denoting the canonical process on $\mathbf{D}^d$ as μ, we define $\mathbb{F}^\mu = \{\mathcal{F}^\mu_s\}_{s\geq 0}$ as the filtration generated by this process. The previous lemma hints at considering a subset of relaxed controls as follows.

Definition 3.4.9. *Fix $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$. $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_s, (X_s)_s, (\bar{\alpha}_s)_s)$ in $\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$ is a* natural control*, and we say that $\mathcal{C}$ is in $\mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$, if $\Omega = \boldsymbol{D}^d$, $\mathcal{F} = \mathcal{F}^\mu_T$, $\mathcal{F}_s = \mathcal{F}^\mu_s$ for $s \in [t,T]$, $X = \mu$, and*

$$\mathbb{P}\left(\mu_s = \lambda, s \in [0,t]\right) = 1.$$

We observe that the pair $(\mathbb{P}, \bar{\alpha})$ determine natural controls, consisting in a probability measure on $\mathbf{D}^d$, *i.e.*, the distribution of μ, and the control process $(\bar{\alpha}_s)_s$. With abuse of notation, we use $(\mathbb{P}, \bar{\alpha})$ to refer to $\mathcal{C}^{\mathbb{P},\bar{\alpha}} := (\mathbf{D}^d, \mathcal{F}^\mu_T, \mathbb{P}, \{\mathcal{F}^\mu_s\}_s, (\mu_s)_s, (\bar{\alpha}_s)_s)$ in $\mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$.

3.4.1 Weak controls

Considering the implications highlighted in Remark 3.3.6, we can focus on a subset of controls known as weak controls. Notably, the elements within this class exhibit uniqueness in terms of their probability distributions. This particular property serves as the crucial connection for identifying the class of strong controls within the realm of relaxed controls.

For a fixed $\mathbf{x} \in \mathbf{D}^d$, the set of measurable functions $\mathfrak{a} : [0,T] \times \mathbb{R}^d \to A$ is canonically embedded in $\mathcal{A}^{\mathrm{Leb},\mathbf{x},1}$ by $\alpha^{\mathfrak{a}}(ds,dx,da) := ds\mathbf{x}_s(dx)\delta_{\mathfrak{a}(s,x)}(da)$.

Definition 3.4.10. *Fix $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$. We say that $(\mathbb{P}, \mathfrak{a})$ is a* weak control*, and we write $(\mathbb{P}, \mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$, if $\mathfrak{a} : [0,T] \times \mathbb{R}^d \times \Omega \to A$ is $\hat{\mathbb{F}}^\mu$-predictable, and $(\mathbb{P}, \alpha^{\mathfrak{a}}) \in \mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$.*

Therefore, for $\mathbb{P} \in \mathcal{R}^0_{(t,\lambda)}$, we have that

$$F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mathfrak{a}(u,x), \mu_u)\mu_u(dx)du$$

is a $(\mathbb{P}, \mathbb{F}^\mu)$-martingale for $s \geq t$, $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$.

We now prove how to restrict the class of controls from $\mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$ to $\mathcal{R}^0_{(t,\lambda)}$ without impacting the value function. This is done by showing that we can always associate natural and weak control with the same cost under the following assumption.

Assumption A9. *The following set*

$$K(x,\lambda) := \Big\{ \Big(b(x,\lambda,a), \sigma\sigma^\top(x,\lambda,a), \big((\gamma p_k)(x,\lambda,a)\big)_{k\geq 0}, z \Big) : a \in A, z \geq \psi(x,\lambda,a) \Big\} \subseteq \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^\infty_+ \times \mathbb{R}$$

is convex for all $(x,\lambda) \in \mathbb{R}^d \times M^1(\mathbb{R}^d)$.

This convexity assumption is the so-called *Filippov condition*, common in the control literature. It holds, for example, when A is a convex subset of a vector space, and the parameters are affine in a, which is the case of the Linear-Quadratic example presented in Section 3.6.3.

Proposition 3.4.16. *Fix $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$. Suppose that Assumption A9 holds. For $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$, there exists $\mathfrak{a}$ such that $(\mathbb{P}, \mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$ and $J\left(t,\lambda; \mathcal{C}^{\mathbb{P},\delta_{\mathfrak{a}}}\right) \geq J\left(t,\lambda; \mathcal{C}^{\mathbb{P},\bar{\alpha}}\right)$.*

Proof. Given $(\mathbb{P}, (\alpha_s)_s)$ in $\mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$, we define c by

$$\begin{aligned} c^1(s,x,\lambda,\omega) &= \int_A \left(b, \sigma\sigma^\top, (\gamma p_k)_{k\geq 0}\right)(x,\lambda,a)\bar{\alpha}_s(x,da), \\ c^2(s,x,\lambda,\omega) &= \int_A \psi(x,\lambda,a)\bar{\alpha}_s(x,da). \end{aligned}$$

All the functions defining K are continuous, therefore, for almost all $(x,\lambda) \in \mathbb{R}^d \times M^1(\mathbb{R}^d)$, $K(x,\lambda)$ is closed. Since $K(x,\lambda)$ is closed and convex, $(c^1,c^2)(s,x,\lambda,\omega)$ is in $K(x,\lambda)$ for any (x,λ) and almost all (s,ω). Moreover, from [86, Lemma A.1], we can take (c^1,c^2) to be $\hat{\mathbb{F}}^\mu$-predictable. We apply [86, Theorem A.9] and obtain that there is a $\hat{\mathbb{F}}^\mu$-predictable A-valued process $\mathfrak{a}$ such that

$$c^1(s,x,\lambda,\omega) = \left(b, \sigma\sigma^\top, (\gamma p_k)_{k\geq 0}\right)(x,\lambda,\mathfrak{a}(s,x,\lambda,\omega)), \tag{3.4.33}$$

$$c^2(s,x,\lambda,\omega) \geq \psi(x,\lambda,\mathfrak{a}(s,x,\lambda,\omega)) \tag{3.4.34}$$

for any (x,λ) and for almost all (s,ω). For $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, we must have

$$\int_{\mathbb{R}^d\times A} \mathcal{L}F_\varphi(x,\mu_u,a_u)\bar{\alpha}_u(x,da)\mu_u(dx) = \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mathfrak{a}(s,x,\mu_u),\mu_u)\mu_u(dx)$$

for almost all (s,ω). Hence, $F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d\times A} \mathcal{L}F_\varphi(x,\mathfrak{a}(s,x,\mu_u),\mu_u)\mu_u(dx)du$ is a martingale, for all $s \geq t$. Therefore, $(\mathbb{P},\mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$, and, from (3.4.33), we get $J(t,\lambda;\mathcal{C}^{\mathbb{P},\delta_{\mathfrak{a}}}) \leq J(t,\lambda;\mathcal{C}^{\mathbb{P},\bar{\alpha}})$. □

3.4.2 Uniqueness in law for weak controls

We introduce the domain $\mathcal{D}$ as the set of function $h:\ \mathbb{R}_+\times\mathbf{D}^d\to\mathbb{R}$ of the form

$$h(s,\mathbf{x}) = F\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_p(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right),\quad (s,\mathbf{x})\in\mathbb{R}_+\times\mathbf{D}^d,$$

for some $p\geq 1$, $0\leq t_1<\cdots<t_p\leq T$, $F\in C_b^2(\mathbb{R}^p)$, and $f_1,\ldots,f_p\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. For $f\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, we use the notation $Lf(s,x,\mu,a)=Lf(s,\cdot)(x,\mu,a)$. For a measurable function $\mathfrak{a}:\mathbb{R}^d\to A$, define the operator $\mathbb{L}^{\mathfrak{a}}$ on $\mathcal{D}$ by

$$\begin{aligned}
\mathbb{L}^{\mathfrak{a}}h(s,\mathbf{x}) \quad = \quad & DF\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right)^\top\mathfrak{L}^{\mathfrak{a}}\mathbf{f}(s,\mathbf{x})\\
&+\frac{1}{2}\mathrm{Tr}\left(\left\langle\mathfrak{S}^{\mathfrak{a}}\mathbf{f}(\mathfrak{S}^{\mathfrak{a}}\mathbf{f})^\top(s,\cdot),\mathbf{x}_s\right\rangle D^2F\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right)\right)\\
&+\sum_{j=1}^p\mathbb{1}_{t_{j-1}<s\leq t_j}\int_{\mathbb{R}^d}\sum_{k\geq 0}\gamma(x,\mathfrak{a}(s,x),\mathbf{x}_s)p_k(x,\mathfrak{a}(s,x),\mathbf{x}_s)\\
&\quad\Big(F\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_{j-1}(s\wedge t_{j-1},\cdot),\mathbf{x}_{s\wedge t_{j-1}}\rangle,\mathfrak{G}_k^1 f_j(s,x,\mathbf{x}_s),\ldots,\mathfrak{G}_k^1 f_p(s,x,\mathbf{x}_s)\right)\\
&\qquad -F\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right)\Big)\mathbf{x}_s(dx)
\end{aligned}$$

with $t_0=0$, where

$$\begin{aligned}
\mathfrak{L}^{\mathfrak{a}}\mathbf{f}(s,\mathbf{x}) \quad &:= \quad \begin{pmatrix}\mathbb{1}_{s\leq t_1}\int_{\mathbb{R}^d}\partial_t f_1(s,x)+Lf_1(s,x,\mathbf{x}_s,\mathfrak{a}(s,x))\mathbf{x}_s(dx)\\ \vdots\\ \mathbb{1}_{s\leq t_p}\int_{\mathbb{R}^d}\partial_t f_p(s,x)+Lf_p(s,x,\mathbf{x}_s,\mathfrak{a}(s,x))\mathbf{x}_s(dx)\end{pmatrix},\\
\mathfrak{S}^{\mathfrak{a}}\mathbf{f}(s,x,\mathbf{x}) \quad &:= \quad \begin{pmatrix}\mathbb{1}_{s\leq t_1}\left|Df_1(s,x)\sigma(x,\mathbf{x}_s,\mathfrak{a}(s,x))\right|\\ \vdots\\ \mathbb{1}_{s\leq t_p}\left|Df_p(s,x)\sigma(x,\mathbf{x}_s,\mathfrak{a}(s,x))\right|\end{pmatrix},\\
\mathfrak{G}_k^n f_j(s,x,\mathbf{x}) \quad &:= \quad \langle f_j(s,\cdot),\mathbf{x}_s\rangle+\frac{k-1}{n}f_j(s,x),
\end{aligned}$$

for $(s,x,\mathbf{x})\in[0,T]\times\mathbb{R}^d\times\mathbf{D}^d$, and $k,j,n\geq 0$.

Considering the canonical process $\mu\in\mathbf{D}^d$, we take the extended process $\mathfrak{x}$ defined by

$$\mathfrak{x}_s \quad = \quad (s,(\mu_{u\wedge s})),\quad s\in[t,T],$$

valued in $\mathbb{R}\times\mathbf{D}^d$, which is separable. Note that for $(\mathbb{P},\mathfrak{a})\in\mathcal{R}^0_{(t,\lambda)}$ the process

$$h(\mathfrak{x}_s)-\int_t^s\mathbb{L}^{\mathfrak{a}}h(\mathfrak{x}_u)du,\quad t\leq u\leq T, \tag{3.4.35}$$

is a $\mathbb{F}^\mu$-martingale under $\mathbb{P}$. Therefore, we have that this condition gives information about the marginals.

Proposition 3.4.17. *Fix $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$ and $(\mathbb{P},\mathfrak{a})\in\mathcal{R}^0_{(t,\lambda)}$. For any $(\mathbb{P}',\mathfrak{a})\in\mathcal{R}^0_{(t,\lambda)}$, $\mathbb{P}$ and $\mathbb{P}'$ have the same one dimensional marginals:*

$$\mathbb{P}(\mathfrak{x}_s\in B) \quad = \quad \mathbb{P}'(\mathfrak{x}_s\in B) \tag{3.4.36}$$

for $s \in [t,T]$ *and* $B \in \mathcal{B}([0,T] \times \boldsymbol{D}^d)$.

Proof. We first endow the measurable space $(\mathbf{D}^d \times \mathbf{D}^d, \mathcal{F}_T^\mu \otimes \mathcal{F}_T^\mu)$ with the probability measure $\mathbb{Q} = \mathbb{P} \otimes \mathbb{P}'$. For $h \in \mathcal{D}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[h \otimes h(\mathfrak{x}_s, \mathfrak{x}_t)\right] \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[h \otimes h(\mathfrak{x}_t, \mathfrak{x}_s)\right]$$

Indeed, the processes

$$h \otimes h(\mathfrak{x}_s, \mathfrak{x}_t) - \int_t^s \mathbb{L}^{\mathfrak{a}} h(\mathfrak{x}_u) h(\mathfrak{x}_t) du, \quad t \le s \le T$$

and

$$h \otimes h(\mathfrak{x}_t, \mathfrak{x}_s) - \int_t^s h(\mathfrak{x}_t) \mathbb{L}^{\mathfrak{a}} h(\mathfrak{x}_u) du, \quad t \le s \le T$$

are both martingales under $\mathbb{Q}$. Since all the considered functions are bounded, we can take the expectation and get

$$\mathbb{E}^{\mathbb{Q}}\left[h \otimes h(\mathfrak{x}_t, \mathfrak{x}_s)\right] \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[h \otimes h(\mathfrak{x}_s, \mathfrak{x}_t)\right]$$

and

$$\mathbb{E}^{\mathbb{P}}\left[h(\mathfrak{x}_s)\right] \quad = \quad \mathbb{E}^{\mathbb{P}'}\left[h(\mathfrak{x}_s)\right].$$

Since any bounded $\mathcal{B}(\mathfrak{X})$-measurable function can be approximated almost everywhere for $\mathbb{P}$ and $\mathbb{P}'$ by a sequence of $\mathcal{D}$ we get (3.4.36). ☐

Theorem 3.4.11. *Fix* $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$ *and* $\mathfrak{a}$ *a* $\hat{\mathbb{F}}^\mu$*-predictable process from* $[0,T] \times \mathbb{R}^d$ *to* A. *There exists at most one* $\mathbb{P} \in \mathcal{P}^1(\boldsymbol{D}^d)$ *such that* $(\mathbb{P}, \mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$, *and we denote it* $\mathbb{P}^{\mathfrak{a}}$.

Proof. The proof is a direct consequence of [74, Theorem 4.2, Chapter 4] and Proposition 3.4.17. ☐

3.4.3 Equivalence between relaxed and strong formulations

Proposition 3.4.18. *Fix* $(t,\lambda) \in [0,T] \times \mathcal{N}[\mathbb{R}^d]$. *For* $\mathfrak{a}$ *a* $\hat{\mathbb{F}}^\mu$*-predictable process from* $[0,T] \times \mathbb{R}^d$ *to* A, *there exist* $\beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$ *and* $\mathbb{P}^{\mathfrak{a}} \in \mathcal{P}^1(\boldsymbol{D}^d)$ *such that* $(\mathbb{P}^{\mathfrak{a}}, \mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$, *and the law of* ξ^β *under* $\mathbb{P}^{\mathfrak{s}}$ *is the same of the one on* μ *under* $\mathbb{P}^{\mathfrak{a}}$.

Proof. Since $\mathfrak{a}$ is $\hat{\mathbb{F}}^\mu$-predictable, from Doob's functional representation theorem (see, *e.g.*, Lemma 1.13 in [100]), there exists a $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{D}^d)$-measurable function $\kappa^{\mathfrak{a}} : [0,T] \times \mathbb{R}^d \times \mathbf{D}^d \to A$ such that $\mathfrak{a}(s,x,\omega) = \kappa^{\mathfrak{a}}(s,x,\mu(\omega_{.\wedge s})) = \kappa^{\mathfrak{a}}(s,x,\mu(\omega))$ for any $s \in [0,T]$, $x \in \mathbb{R}^d$, and $\omega \in \Omega$.

Fix some $a_0 \in A$. We consider the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as in Section 3.2 and define the standard strong control $\beta^{\mathfrak{a}}$ as

$$\beta_s^{\mathfrak{a},i} = \kappa^{\mathfrak{a}}\left(s, Y_s^{i,\beta}, \left(\xi^\beta_{u \wedge s}\right)_{u \in [0,T]}\right) \mathbb{1}_{i \in V_s} + a_0 \mathbb{1}_{i \notin V_s},$$

where ξ^β (resp. $Y^{i,\beta}$ for $i \in V_s$) is the strongly controlled population (resp. particle) associated with $\beta^{\mathfrak{a}}$. From Proposition 3.2.13, there exists a unique càdlàg process that satisfies (3.2.6) associated with this control $\beta^{\mathfrak{a}}$. Moreover, condition (3.2.17) is satisfied, hence $\beta^{\mathfrak{a}} \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$.

With the embedding given in Remark 3.3.6, we can associate to $\beta^{\mathfrak{a}}$ the relaxed control

$$\mathcal{C}^{\mathfrak{a}} = (\Omega^{\mathfrak{a}}, \mathcal{F}^{\mathfrak{a}}, \mathbb{P}^{\mathfrak{a}}, \{\mathcal{F}^{\mathfrak{a}}_s\}_s, (X^{\mathfrak{a}}_s)_s, (\bar{\alpha}^{\mathfrak{a}}_s)_s).$$

From Lemma 3.4.6, we get a natural control $(\mathbb{P}^{\mathfrak{n},\mathfrak{a}}, \bar{\alpha}^{\mathfrak{n},\mathfrak{a}})$. Following [86, Lemma 3.7], since $\bar{\alpha}^{\mathfrak{a}}$ is a Dirac measure $\mathbb{P}^{\mathfrak{a}}$-a.s., we have that $\bar{\alpha}^{\mathfrak{n},\mathfrak{a}}$ is a Dirac measures $\mathbb{P}^{\mathfrak{n},\mathfrak{a}}$-a.s. Moreover, we can see that $\bar{\alpha}^{\mathfrak{n},\mathfrak{a}}_s(x, da) = \delta_{\kappa^{\mathfrak{a}}(s,x,\mu(\omega_{\cdot\wedge s}))} = \delta_{\mathfrak{a}(s,x)}$ $\mathbb{P}^{\mathfrak{n},\mathfrak{a}}$-a.s., hence $(\mathbb{P}^{\mathfrak{n},\mathfrak{a}}, \mathfrak{a}) \in \mathcal{R}^0_{(t,\lambda)}$. □

Combining Theorem 3.4.11 and Proposition 3.4.18, we have that a weak control is specified by the $\hat{\mathbb{F}}^{\mu}$-predictable control $\mathfrak{a}$. With abuse of notation, we write $\mathfrak{a} \in \mathcal{R}^0_{(t,\lambda)}$ (resp. $J(t, \lambda; \mathfrak{a})$) to denote $\mathcal{C}^{\mathfrak{a}} := \big(\mathbf{D}^d, \mathcal{F}^{\mu}_T, \mathbb{P}^{\mathfrak{a}}, \{\mathcal{F}^{\mu}_s\}_s, (\mu_s)_s, \big(\delta_{\mathfrak{a}(s,\cdot)}\big)_s\big) \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$ (resp. $J(t, \lambda; \mathcal{C}^{\mathbb{P}})$).

Proposition 3.4.19. *Suppose Assumption A9 holds. For $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, we have*

$$\begin{aligned} v(t, \lambda) := \inf\Big\{J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}\Big\} &= \inf\Big\{J(t, \lambda; \mathfrak{a}) : \mathfrak{a} \in \mathcal{R}^0_{(t,\lambda)}\Big\} \\ &= \inf\Big\{J(t, \lambda; \beta) : \beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}\Big\} \ . \end{aligned}$$

Proof. We denote $v^r(t, \lambda) = \inf\Big\{J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}\Big\}$, $v^0(t, \lambda) = \inf\Big\{J(t, \lambda; \mathfrak{a}) : \mathfrak{a} \in \mathcal{R}^0_{(t,\lambda)}\Big\}$ and $v^s(t, \lambda) = \inf\Big\{J(t, \lambda; \beta) : \beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}\Big\}$. From the embedding of Remark 3.3.6, we have that $v^r(t, \lambda) \leq v^s(t, \lambda)$. Using Lemma 3.4.6 and Proposition 3.4.16, for each relaxed control, there exists a weak control that does not increase the value functions. This means that $v^r(t, \lambda) = v^0(t, \lambda)$. Finally, from Proposition 3.4.18, any weak control finds a representation in the strong controls set. This means that $v^s(t, \lambda) \leq v^0(t, \lambda)$. □

We can now give the bounds on the moments of the controlled processes in the relaxed framework.

Proposition 3.4.20. *Let $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, and*

$$\mathcal{C} = \Big(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_{s\geq 0}, (X_s)_{s\geq 0}, (\bar{\alpha}_s)_{s\geq 0}\Big) \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}.$$

There exists a constant $C > 0$ depending only on T and on the coefficients b, σ, γ and $(p_k)_k$

such that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,t+h]}\langle 1,X_u\rangle\right] \leq \langle 1,\lambda\rangle e^{C_\gamma C_\Phi^1 h}, \tag{3.4.37}$$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,t+h]}\langle 1,X_u\rangle^2\right] \leq \langle 1,\lambda\rangle e^{C_\gamma (C_\Phi^1+C_\Phi^2) h}, \tag{3.4.38}$$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,t+h]}\langle|\cdot|,X_u\rangle\right] \leq C\bigg(\langle|\cdot|,\lambda\rangle + \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\langle 1,X_u\rangle du\right] + \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\int_{\mathbb{R}^d\times A}|a|\bar{\alpha}_u(x,da)X_u(dx)du\right]\bigg), \tag{3.4.39}$$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,t+h]}\langle|\cdot|^2,X_u\rangle\right] \leq C\bigg(\langle|\cdot|^2,\lambda\rangle + \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\langle 1,X_u\rangle du\right] + \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\int_{\mathbb{R}^d\times A}|a|^2\bar{\alpha}_u(x,da)X_u(dx)du\right]\bigg), \tag{3.4.40}$$

for any $h>0$, where $|\cdot|$ (resp. $|\cdot|^2$) denote the function $x\mapsto|x|$ (resp. $x\mapsto|x|^2$). Moreover, for $\varepsilon>0$, if $\mathcal{R}^{\mathfrak{r},\varepsilon}_{(t,\lambda)}$ denotes the set of $\mathcal{C}\in\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$ satisfying $J(t,\lambda;\mathcal{C})\leq v(t,\lambda)+\varepsilon$. Then

$$\sup_{\beta\in\mathcal{R}^{\mathfrak{r},\varepsilon}_{(t,\lambda)}}\mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\int_{\mathbb{R}^d\times A}|a|^2\bar{\alpha}_u(x,da)X_u(dx)du\right] < \infty. \tag{3.4.41}$$

Proof. From Lemma 3.4.6, any bound established on relaxed control transposes exactly on natural controls. Fix $(\mathbb{P},(\alpha_s)_s)\in\mathcal{R}^{\mathfrak{n}}_{(t,\lambda)}$. From the proof of Proposition 3.4.16, we see that the weak control $(\mathbb{P},\mathfrak{a})\in\mathcal{R}^0_{(t,\lambda)}$ associated with this natural control does not modify the probability measure $\mathbb{P}$, nor the law of μ, using Assumption A9. In particular, this procedure can be applied for any kind of cost functions (ψ,Ψ) as soon as they satisfy the bounds (3.2.13)-(3.2.16).

Define now $\psi^1(x,\lambda,a) := |a|$ (resp. $\psi^2(x,\lambda,a) := |a|^2$). Since ψ^1 (resp. ψ^2) satisfies (3.2.13)-(3.2.16), we consider $\mathfrak{a}^1$ (resp. $\mathfrak{a}^2$) the weak control associated with the couple $(\psi^1,0)$ (resp. $(\psi^2,0)$). In the notation of the paper, the cost functions associated with these couples are respectively

$$J_p(t,\lambda;\mathcal{C}) = \mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_{\mathbb{R}^d\times A}|a|^p\bar{\alpha}_s(x,da)\mu_s(dx)ds\right], \quad \text{for } p=1,2.$$

Using the identification between weak, controls and strong controls, we have that (3.2.7), (3.2.8), (3.2.10), and (3.2.19) extend directly to the framework of weak controls. Therefore, since the first two depend only on the parameters of the model and the initial condition (t,λ), we get (3.4.37) and (3.4.38).

Since the association from α to $\mathfrak{a}^1$ (resp. $\mathfrak{a}^2$) given by Proposition 3.4.16 is non-increasing in the cost function, we have that

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_{\mathbb{R}^d}|\mathfrak{a}^1(s,x)|\mu_s(dx)ds\right] \le \mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_{\mathbb{R}^d\times A}|a|\bar{\alpha}_s(x,da)\mu_s(dx)ds\right],$$
$$\mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_{\mathbb{R}^d}|\mathfrak{a}^2(s,x)|^2\mu_s(dx)ds\right] \le \mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_{\mathbb{R}^d\times A}|a|^2\bar{\alpha}_s(x,da)\mu_s(dx)ds\right].$$

Therefore, combining these inequalities with (3.2.10) and (3.2.19), we get exactly (3.4.39) and (3.4.40).

Finally, to retrieve (3.4.41), we argue exactly as in Proposition 3.2.14 directly in the relaxed control setting. This is again a consequence that the function $a \mapsto |a|^2 - C|a|$ is bounded below and (3.2.13)-(3.2.16).

□

3.5 Existence of Optimal Controls

We look for canonic relaxed controls to show the existence of optimal controls. From Lemma 3.4.6, we can define the control problem 3.3.31-3.3.32 with respect to any class $\mathcal{R}^{\cdot}$ such that $\mathcal{R}^{\mathfrak{n}} \subseteq \mathcal{R}^{\cdot} \subseteq \mathcal{R}^{\mathfrak{r}}$ without increasing the value function. Since we focus on the pair (X,α) in the definition of relaxed controls, canonic relaxed controls are defined in $\Omega = \mathbf{D}^d \times \mathcal{A}^{\mathrm{Leb},\cdot,1}$. Let $(\mu,\mathbf{a})$ be the projection maps (or canonical processes) on $\mathbf{D}^d \times \mathcal{A}^{\mathrm{Leb},\cdot,1}$, and $\mathbb{F}^{\mu,\mathbf{a}} = \{\mathcal{F}_s^{\mu,\mathbf{a}}\}_s$ the filtration generated by them, *i.e.*,

$$\sigma\left(\mu_s(B_1), \mathbf{a}([0,s']\times B_2\times B_3],\ \text{for } s,s'\in[0,T], B_1,B_2\in\mathcal{B}(\mathbb{R}^d), B_3\in\mathcal{B}(A)\right).$$

Definition 3.5.11 (Control rule)**.** *Fix* $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$. $\mathcal{C} = (\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_s\}_s,(X_s)_s,(\bar{\alpha}_s)_s) \in \mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}$ *is a* control rule*, and we write* $\mathcal{C}\in\mathcal{R}_{(t,\lambda)}$*, if* $\Omega = \boldsymbol{D}^d\times\mathcal{A}^{Leb,\cdot,1}$, $\mathcal{F} = \mathcal{F}_T^{\mu,\mathbf{a}}$, $\mathcal{F}_s = \mathcal{F}_s^{\mu,\mathbf{a}}$ *for* $s\in[t,T]$, $X=\mu$, $\alpha=\mathbf{a}$ *and*

$$\mathbb{P}\left(\mu_s = \lambda, s\in[0,t]\right) = 1.$$

A control rule is specified by $\mathbb{P}\in\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$, *i.e.*, the distribution of $(\mu,\mathbf{a})$. With abuse of notation, we write $\mathbb{P}\in\mathcal{R}_{(t,\lambda)}$ (resp. $J(t,\lambda;\mathbb{P})$) to denote $\mathcal{C}^{\mathbb{P}} := (\mathbf{D}^d,\mathcal{F}_T^{\mu},\mathbb{P},\{\mathcal{F}_s^{\mu}\}_s,(\mu_s)_s,(\bar{\mathbf{a}}_s)_s)\in\mathcal{R}_{(t,\lambda)}$ (resp. $J(t,\lambda;\mathcal{C}^{\mathbb{P}})$).

From Lemma 3.4.6, any relaxed control is associated with a control rule with the same cost function J. Therefore,

$$v(t,\lambda) = \inf\left\{J(t,\lambda;\mathcal{C}) : \mathcal{C}\in\mathcal{R}^{\mathfrak{r}}_{(t,\lambda)}\right\} = \inf\left\{J(t,\lambda;\mathbb{P}) : \mathbb{P}\in\mathcal{R}_{(t,\lambda)}\right\}.$$

We aim to apply the same procedure, as in [86] and [107], to exhibit the existence of a relaxed control. This means proving the optimization problem consists of minimizing a lower semicontinuous function on a compact set. Therefore, we aim to show that J is lower semicontinuous and $\mathcal{R}^{\varepsilon}_{(t,\lambda)} := \mathcal{R}^{\mathfrak{r},\varepsilon}_{(t,\lambda)}\cap\mathcal{R}_{(t,\lambda)}$ is compact in $\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$ for $\varepsilon>0$.

Lemma 3.5.7. *For* $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$, $J(t,\lambda;\cdot)$ *is lower semicontinuous on* $\mathcal{P}^1(\boldsymbol{D}^d\times\mathcal{A}^{Leb,\cdot,1})$.

Proof. Consider $f : \mathbf{D}^d \times \mathcal{A}^{\text{Leb},\cdot,1} \to \mathbb{R}$, defined as

$$f(\mathbf{x}, \alpha) := \int_t^T \int_{\mathbb{R}^d \times A} \psi(x, \mathbf{x}_s, a)\, \bar{\alpha}_s(x, da)\mathbf{x}_s(dx)ds + \Psi(\mathbf{x}_T).$$

This function is lower semicontinuous as a consequence of the continuity of ψ and Ψ and their growth conditions (3.2.16) and (3.2.14). This means that $J(t, \lambda; \mathbb{P}) = \int f d\mathbb{P}$ is lower semicontinuous. □

For a Polish space (E, d) and $\mathbb{P} \in \mathcal{P}(M(E))$, we define the mean measure $m\mathbb{P} \in \mathcal{P}(E)$ by

$$m\mathbb{P}(C) := \int_{M(E)} \lambda(C)\mathbb{P}(d\lambda).$$

Since $\mathrm{d}_{p,E}$ is a Wasserstein type distance, from (3.2.1), the results from [108, Appendix B] can be naturally extended to this setting. As the primary focus is on convergence in weak* topology in the first part, we will examine an alternative metrization, simpler than $\mathrm{d}_{p,E}$.

A family $\mathscr{F} \subseteq C_b(E)$ is said to be *separating* if, whenever $\langle \varphi, \lambda \rangle = \langle \varphi, \lambda' \rangle$ for all $\varphi \in \mathscr{F}$, and some $\lambda, \lambda' \in M(E)$, we necessarily have $\lambda = \lambda'$. Since E is Polish, from the Portmanteau theorem (see, *e.g.*, [148, Theorem 1.1.1]), the set of uniformly continuous functions, for any metric equivalent to d, is separating. Using Tychonoff's embedding theorem (see, *e.g.*, [154, Theorem 17.8]), $C_b(E)$ is also separable. Therefore, there exists a countable and separating family $\mathscr{F}_E = \{\varphi_k, k \in \mathbb{N}\}$ subset of $C_b(E)$ such that the function $E \ni x \mapsto 1$ belongs to $\mathscr{F}_E$ and $||\varphi_k||_\infty := \sup_E |\varphi_k| \leq 1$ for all $k \in \mathbb{N}$ since multiplying by a positive constant do not impact the property of being separating. With the use of this family,

$$\mathrm{d}_{\text{weak*},E}(\lambda, \lambda') = \sum_{\varphi_k \in \mathscr{F}_E} \frac{1}{2^k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|,$$

for $\lambda, \lambda' \in M(E)$. As in [148, Theorem 1.1.2], this distance $\mathrm{d}_{\text{weak},E}$ induces on $M(E)$ the weak* topology. Whenever $E = \mathbb{R}^d$, we can adjust this metric to take into account useful differential properties. Let $\mathscr{F}_{\mathbb{R}^d}$ be taken as a subset of $C_b^2(\mathbb{R}^d)$, the set of real functions with bounded, continuous derivatives over $\mathbb{R}^d$ up to order two. Without loss of generality, since C^2 is dense in C^0, this set is separating under local uniform convergence (application of [82, Theorem 8.14]). Moreover, since $\mathbf{x} \mapsto 1$ belongs to $\mathscr{F}_{\mathbb{R}^d}$, adding a constant or multiplying by a non-negative constant to each function does not change the property of being a separating set, we assume $\varphi_k \geq 0$. We define the distance

$$\mathrm{d}_{\text{weak*},\mathbb{R}^d}(\lambda, \lambda') = \sum_{\varphi_k \in \mathscr{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|, \tag{3.5.42}$$

with $q_k = \max\{1, ||D\varphi_k||_\infty, ||D^2\varphi_k||_\infty\}$.

Proposition 3.5.21. *Given $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ and $\varepsilon > 0$, $\mathcal{R}^\varepsilon_\lambda$ is compact in $\mathcal{P}^1(\boldsymbol{D}^d \times \mathcal{A}^{Leb,\cdot,1})$.*

Proof. The proof of this lemma breaks into four steps.

Step 1. First, we aim to prove that $\left\{ m\mathbb{P}_{|\mathbf{D}^d} : \mathbb{P} \in \mathcal{R}^\varepsilon_{(t,\lambda)} \right\} \subseteq \mathcal{P}(\mathbf{D}^b)$ is tight. To do that, we verify Aldous criterion (see, *e.g.*, [100, Theorem 14.11]), *i.e.*, proving

$$\lim_{\delta \downarrow 0} \sup_{\mathbb{P} \in \mathcal{R}_{(t,\lambda)}} \sup_\tau \mathbb{E}^{\mathbb{P}} \left[\mathrm{d}_{\text{weak*},\mathbb{R}^d}(\mu_{(\tau+\delta)\wedge T}, \mu_\tau) \right] = 0, \tag{3.5.43}$$

where the innermost supremum is over stopping times τ valued in $[t, T]$.

From Proposition 3.3.15, we know there exists an extension $\hat{\Omega}$ of $\mathbf{D}^d \times \mathcal{A}^{\mathrm{Leb},\cdot,1}$ where μ can be represented as the solution of (3.3.29). This SDE is driven by $\mathcal{M}^c$ orthogonal continuous martingale measure on $\hat{\Omega} \times [0,T] \times \mathbb{R}^d \times A$, with intensity measure $ds\mu_s(dx)\bar{\mathbf{a}}_s(x,da)$, and a purely discontinuous martingale measure $\mathcal{M}^d$ on $\hat{\Omega} \times [0,T] \times \mathbb{R}^d \times \mathbb{R}_+ \times A$, with dual predictable projection measure $ds\mu_s(dx)dz\bar{\mathbf{a}}_s(x,da)$. Applying (3.3.29) to $\varphi_k \in \mathscr{F}_{\mathbb{R}^d}$, we get

$$\begin{aligned}
\langle \varphi_k, \mu_{(s+\delta)\wedge T}\rangle = \langle \varphi_k, \mu_s\rangle \quad &+ \quad \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} \big(L\varphi_k(x,\mu,a_r) + \\
&\qquad \gamma(x,\mu,a_r)\left(\partial_s \Phi(1,x,\mu,a_r) - 1\right)\varphi_k(x)\big)\bar{\mathbf{a}}_r(x,da)\mu_r(dx)dr + \\
&+ \quad \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} D\varphi_k(x)\sigma(x,X_r,a)\mathcal{M}^c(dr,dx,da) \\
&+ \quad \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k\geq 0} \langle \varphi_k, (k-1)\delta_x\rangle \mathbb{1}_{I_k(x,\mu_r,a)}(z)\mathcal{M}^d(dr,dx,dz,da).
\end{aligned}$$

for $s \in [0,T]$, $k \in \mathbb{N}$. Therefore, to bound the quantity $\mathbb{E}^{\mathbb{P}}\left[|\langle \varphi_k, \mu_{(s+\delta)\wedge T}\rangle - \langle \varphi_k, \mu_s\rangle|\right]$, it suffices to bound the last three terms in the r.h.s. There is a constant $C > 0$ that depends only on b, σ, γ, and Φ (which may change from line to line) such that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}&\left[\left| \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} \big(L\varphi_k(x,\mu,a_r) + \gamma(x,\mu,a_r)\left(\partial_s\Phi(1,x,\mu,a_r) - 1\right)\varphi_k(x)\big)\bar{\mathbf{a}}_r(x,da)\mu_r(dx)dr\right|\right] \leq \\
&\leq Cq_k \mathbb{E}^{\mathbb{P}}\left[\int_s^{(s+\delta)\wedge T} \left(\langle 1, \mu_u\rangle + \langle |\cdot|, \mu_u\rangle\right) du + \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} |a| \bar{\mathbf{a}}_u(x,da)\mu_u(dx)du\right].
\end{aligned}$$

Applying Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\left|\int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} D\varphi_k(x)\sigma(x,X_r,a)\mathcal{M}^c(dr,dx,da)\right|\right] \leq Cq_k \mathbb{E}^{\mathbb{P}}\left[\int_s^{(s+\delta)\wedge T} \langle 1, \mu_u\rangle du\right].$$

Finally, since $\varphi_k \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\left|\int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k\geq 0} \langle \varphi_k, (k-1)\delta_x\rangle \mathbb{1}_{I_k(x,\mu_r,a)}(z)\mathcal{M}^d(dr,dx,dz,da)\right|\right] \\
&\quad \leq \mathbb{E}^{\mathbb{P}}\left[\left|\int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} \varphi_k(x) \sum_{k\geq 1} (k-1)\gamma\left(x,\mu_r,a\right) p_k\left(x,\mu_r,a\right) \bar{\mathbf{a}}_r(x,da)\mu_r(dx)dr\right|\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq Cq_k \mathbb{E}^{\mathbb{P}}\left[\int_s^{(s+\delta)\wedge T} \langle 1, \mu_u\rangle du\right].
\end{aligned}$$

Combining these inequalities, we get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[|\langle\varphi_k,\mu_{(s+\delta)\wedge T}\rangle-\langle\varphi_k,\mu_s\rangle|\right]\\
&\le Cq_k\mathbb{E}^{\mathbb{P}}\left[\int_s^{(s+\delta)\wedge T}(\langle 1,\mu_u\rangle+\langle|\cdot|,\mu_u\rangle)\,du+\int_s^{(s+\delta)\wedge T}\int_{\mathbb{R}^d\times A}|a|\bar{\mathbf{a}}_u(x,da)\mu_u(dx)du\right]\\
&\le \delta Cq_k\left(\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[0,T]}(\langle 1,\mu_u\rangle+\langle|\cdot|,\mu_u\rangle)\right]+\mathbb{E}^{\mathbb{P}}\left[\int_s^{(s+\delta)\wedge T}\int_{\mathbb{R}^d\times A}|a|\bar{\mathbf{a}}_u(x,da)\mu_u(dx)du\right]\right).
\end{aligned}$$

Combining (3.4.37) and (3.4.40), together with the uniform bound (3.4.41), we obtain $\mathbb{E}^{\mathbb{P}}\left[|\langle\varphi_k,\mu_{(s+\delta)\wedge T}\rangle-\langle\varphi_k,\mu_s\rangle|\right]\le Cq_k\delta$. Multiplying for $\frac{1}{2^k q_k}$, summing over $k\in\mathbb{N}$ and applying the monotone convergence theorem, we get $\mathbb{E}^{\mathbb{P}}\left[\mathrm{d}_{\mathbb{R}^d}(\mu_{(s+\delta)\wedge T},\mu_s)\right]\le\delta C$, which gives us (3.5.43).

Step 2. Secondly, we prove that $\left\{\mathbb{P}_{|\mathbf{D}^d}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}\subseteq\mathcal{P}^1(\mathbf{D}^b)$ is relatively compact. Combining the bound (3.4.40) with (3.4.41) and (3.4.37), we get

$$\sup_{\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}}\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,T]}\int_{\mathbb{R}^d}|x|^2\mu_u(dx)\right]<\infty.$$

This bound, together with (3.4.38) and (3.2.1), gives that

$$\sup_{\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}}\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,T]}\mathrm{d}^2_{2,\mathbb{R}^d}(\mu_u,\delta_0)\right]<\infty. \tag{3.5.44}$$

Putting together *Step 1* and this bound, we have from [108, Corollary B.2] that $\left\{\mathbb{P}_{|\mathbf{D}^d}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}\subseteq\mathcal{P}^1(\mathbf{D}^b)$ is relatively compact.

Step 3. From the first step, we have that $\left\{\mathbb{P}\circ\mu^{-1}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}$ is tight in $\mathcal{P}^1(\mathbf{D}^b)$. Adding this to (3.4.41) and (3.5.44), we have that $\left\{\mathbb{P}_{|\mathcal{A}^{\mathrm{Leb},\cdot,1}}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}$ is compact in $\mathcal{P}^1(\mathcal{A}^{\mathrm{Leb},\cdot,1})$. This entails that $\mathcal{R}^{\varepsilon}_{(t,\lambda)}$ is relatively compact in $\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$ since $\left\{\mathbb{P}_{|\mathbf{D}^d}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}$ and $\left\{\mathbb{P}_{|\mathcal{A}^{\mathrm{Leb},\cdot,1}}:\mathbb{P}\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}\right\}$ are relatively compact in $\mathcal{P}^1(\mathbf{D}^d)$ and $\mathcal{P}^1(\mathcal{A}^{\mathrm{Leb},\cdot,1})$ respectively.

Step 4. Finally, we prove $\mathcal{R}^{\varepsilon}_{(t,\lambda)}$ is closed. To do that, we show that $\mathbb{P}^\infty$ belongs to $\mathcal{R}^{\varepsilon}_{(t,\lambda)}$ for $\mathbb{P}^n\to\mathbb{P}^\infty$ in $\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$, with $\mathbb{P}^n\in\mathcal{R}^{\varepsilon}_{(t,\lambda)}$. Since μ_t has law λ under $\mathbb{P}^n$, the same is true under $\mathbb{P}^\infty$. Analogously, since $\mathbb{P}^n(\alpha\in\mathcal{A}^{\mathrm{Leb},\mu,1})=1$, the same is true under $\mathbb{P}^\infty$. For any $F\in C^2_b(\mathbb{R})$ and $\varphi\in C^2_b(\mathbb{R}^d)$ and $\mathbb{P}\in\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$, define $M^{\mathbb{P},F_\varphi}_s:\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1}\to\mathbb{R}$ by

$$M^{\mathbb{P},F_\varphi}_s(\mathbf{x},\alpha)\quad=\quad F_\varphi(\mathbf{x}_s)-\int_t^s\int_{\mathbb{R}^d\times A}\mathcal{L}F_\varphi(y,\mathbf{y}_u,a)\bar{\alpha}_u(y,da)\mathbf{y}_u(dy)\delta_{\mathbf{y}_u=\mathbf{x}_u}du.$$

Recalling the definition of $\mathcal{L}$, we see that there exists a constant $C>0$ depending only on the bounds of F, φ and the constants C_b, C_σ, C_γ such that

$$|\mathcal{L}F_\varphi(y,\lambda,a)|\le C(1+|y|+|a|).$$

This implies

$$\left|M_s^{\mathbb{P},F_\varphi}(\mathbf{x},\alpha)\right| \le C\left(1+\sup_{u\in[t,T]} \mathrm{d}_{1,\mathbb{R}^d}\left(\mathbf{x}_u,\delta_0\right)+\int_t^T\int_{\mathbb{R}^d\times A}|a|\bar{\alpha}_u(x,da)\mathbf{y}_u(dx)du\right).$$

Combining this with the continuity of b, σ, γ and p_k for $k\in\mathbb{N}$, we have that $(\mathbb{P},\mathbf{x},\alpha)\mapsto M_s^{\mathbb{P},F_\varphi}(\mathbf{x},\alpha)$ is a continuous function for each $s\in[t,T]$, $F\in C_b^2(\mathbb{R})$ and $\varphi\in C_b^2(\mathbb{R}^d)$ using [108, Corollary A.5]. Since $\mathbb{P}^n\to\mathbb{P}^\infty$ in $\mathcal{P}^1(\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1})$, it follows that

$$\mathbb{E}^{\mathbb{P}^\infty}\left[\left(M_{s+u}^{\mathbb{P}^\infty,F_\varphi}-M_s^{\mathbb{P}^\infty,F_\varphi}\right)\Lambda\right]=\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}^n}\left[\left(M_{s+u}^{\mathbb{P}^n,F_\varphi}-M_s^{\mathbb{P}^n,F_\varphi}\right)\Lambda\right],$$

for every $s\in[t,T]$, $u\ge 0$ such that $s+u\le T$, any $F\in C_b^2(\mathbb{R})$ and $\varphi\in C_b^2(\mathbb{R}^d)$, and any bounded continuous function Λ on $\mathbf{D}^d\times\mathcal{A}^{\mathrm{Leb},\cdot,1}$, measurable with respect to $\sigma\left(\mu_u,\bar{\mathbf{a}}_u : u\in[t,s]\right)$. Since $\mathbb{P}^n\in\mathcal{R}^\varepsilon_{(t,\lambda)}$, the process $\left(M_s^{\mathbb{P}^n,F_\varphi}(\mu,\mathbf{a})\right)_{s\in[0,T]}$ is a martingale under $\mathbb{P}^n$, and the above quantity is zero. This shows that $\left(M_s^{\mathbb{P}^\infty,\varphi}(\mu,\mathbf{a})\right)_{s\in[0,T]}$ is a martingale under $\mathbb{P}^\infty$, and so $\mathbb{P}^\infty\in\mathcal{R}_{(t,\lambda)}$.

Moreover, by Lemma 3.5.7 we get since J is lower semicontinuous. Therefore,

$$J\left(t,\lambda;\mathbb{P}^\infty\right)\le\liminf_{n\to\infty}J(t,\lambda;\mathbb{P}^n)\le v(t,\lambda)+\varepsilon,$$

which means that $\mathbb{P}^\infty\in\mathcal{R}^\varepsilon_{(t,\lambda)}$. □

Theorem 3.5.12. *For $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$, there exists an optimal control $\beta^*\in\mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$ such that*

$$v(t,\lambda)=J(t,\lambda;\beta^*). \tag{3.5.45}$$

Proof. Fix $\varepsilon>0$. We have that $\inf_{\mathbb{P}\in\mathcal{R}_{(t,\lambda)}}J(t,\lambda;\mathbb{P})=\inf_{\mathbb{P}\in\mathcal{R}^\varepsilon_{(t,\lambda)}}J(t,\lambda;\mathbb{P})$. By Proposition 3.5.21, $\mathcal{R}^\varepsilon_{(t,\lambda)}$ is compact and, by Lemma 3.5.7, J is lower-semicontinuous. Therefore, since $v(t,\lambda)$ is the infimum of a continuous function over a nonempty compact set, it exists $\mathbb{P}^*\in\mathcal{R}_{(t,\lambda)}$ such that $v(t,\lambda)=J(t,\lambda;\mathbb{P}^*)$. From Lemma 3.4.6 and Proposition 3.4.16, under Assumption A9, we have the existence of optimal weak control $\mathfrak{a}^*$ such that $J(t,\lambda;\mathfrak{a}^*)\le J(t,\lambda;\mathbb{P}^*)$. Immerging this weak control in the class of strong controls, we find β^* that satisfies (3.5.45). □

3.6 HJB equation

3.6.1 Homeomorphisms with $\cup_{m\ge 0}\mathbb{R}^{dm}$

We have established the existence of an optimal control for the problem under consideration, which holds true under general assumptions. However, the formalism we have discussed thus far does not provide guidance on how to determine these optimal controls. A step towards addressing this is the differential characterization of the value function, commonly referred to as the Hamilton–Jacobi–Bellman (HJB) equation.

Though the problem has been stated in terms of finite measures, this depiction cannot be employed directly to tackle the task at hand. Indeed, the subset $\mathcal{N}[\mathbb{R}^d]$ where our processes live is not open in $\left(M^1(\mathbb{R}^d),\mathrm{d}_{1,\mathbb{R}^d}\right)$. As recalled in Remark 3.2.4, we can embed $\mathbb{R}^{dm}$ to $\mathcal{N}[\mathbb{R}^d]$ for any $m\in\mathbb{R}^d$ via ι. Denoting $\left(\mathbb{R}^d\right)^0:=\{\emptyset\}$, and $\iota(\emptyset):=\mathbb{O}$, which is the measure equal to 0, we see that $\iota\left(\bigcup_{m\geq 0}\left(\mathbb{R}^d\right)^m\right)=\mathcal{N}[\mathbb{R}^d]$. Therefore, we can define a HJB system exploiting the differential structure of each $\mathbb{R}^{dm}$.

For each $m\in\mathbb{N}$, let $v^m:[0,T]\times\left(\mathbb{R}^d\right)^m\to\mathbb{R}$ be

$$v^m(t,x_1,\dots,x_m):=v\left(t,\sum_{i=1}^m\delta_{x_i}\right)=v\left(t,\iota(\vec{x}^m)\right), \tag{3.6.46}$$

with $\vec{x}^m=(x_1,\dots,x_m)^\top$. Analogously, we define $(\mathfrak{b}^m,\Sigma^m):\left(\mathbb{R}^d\right)^m\times A^m\to\mathbb{R}^{dm}\times\mathbb{R}^{dm\times d'}$ as

$$\mathfrak{b}^m\left(\vec{x}^m,\vec{a}^m\right):=\begin{pmatrix}b\left(x_1,\iota(\vec{x}^m),a_1\right)\\ \vdots\\ b\left(x_m,\iota(\vec{x}^m),a_m\right)\end{pmatrix},\qquad \Sigma^m\left(\vec{x}^m,\vec{a}^m\right):=\begin{pmatrix}\sigma\left(x_1,\iota(\vec{x}^m),a_1\right)\\ \vdots\\ \sigma\left(x_m,\iota(\vec{x}^m),a_m\right)\end{pmatrix}.$$

For any $m\in\mathbb{N}$, let $\mathbf{L}^m$ be the generator as follows

$$\begin{aligned}\mathbf{L}^m v^m\left(\vec{x}^m,\vec{a}^m\right) \quad := \quad & \mathfrak{b}^m\left(\vec{x}^m,\vec{a}^m\right)^\top Dv^m\left(\vec{x}^m\right)+\frac{1}{2}\mathrm{Tr}\left(\Sigma^m(\Sigma^m)^\top\left(\vec{x}^m,\vec{a}^m\right)D^2v^m\left(\vec{x}^m\right)\right)\\ &+\sum_{i=1}^m\gamma\left(x_i,\iota(\vec{x}^m),a_i\right)\left(\sum_{k\geq 0}v^{m+(k-1)}\left(x_1,\dots,x_{i-1},\underbrace{x_i,\dots,x_i}_{(k-1)-\text{times}},x_{i+1},\dots,x_m\right)\right.\\ &\left.\qquad p_k\left(x_i,\iota(\vec{x}^m),a_i\right)-v^m\left(\vec{x}^m\right)\right).\end{aligned}$$

Remark 3.6.7. *These notations look like the one used in Proposition 3.2.13. As seen in their construction, branching processes behave as diffusion processes between two different branching events, that are defined via a Poisson random measure independent of each Brownian motion. This is why the first two terms of $\boldsymbol{L}^m$ are Itô's-like terms while the last one takes into account the results of the branching events.*

Since our aim is giving a Verification Theorem, we associate an admissible control from a set of functions $\hat{a}^m:[0,T]\times\left(\mathbb{R}^d\right)^m\to A^m$ in the following way. As done in [42] and Chapter 1, we consider the partial ordering relation $\preceq$ (resp. $\prec$) by

$$j\preceq i\;\Leftrightarrow\;\exists\ell\in\mathcal{I}:i=j\ell\qquad(\text{resp. } j\prec i\;\Leftrightarrow\;\exists\ell\in\mathcal{I}\setminus\{\varnothing\}:i=j\ell)$$

for all $i,j\in\mathcal{I}$. With respect to this partial ordering, for $i=i_0\dots i_p, j=j_0\dots j_q\in\mathcal{I}$, we define $i\wedge j$ as $\emptyset$ in the case $i_0\neq j_0$, and as $i_0\dots i_{\ell-1}$ with $\ell\leq\min\{p,q\}$ if $j_k=i_k$ for $k=0,\dots,\ell-1$ and $j_k\neq i_k$. If $\mathcal{I}^{\preceq}$ is defined as

$$\mathcal{I}^{\preceq}\quad=\quad\left\{V\subseteq\mathcal{I}:|V|<\infty, i\not\prec j\text{ for }i,j\in V\right\},$$

the set of labels that could describe a population in $\mathcal{N}[\mathbb{R}^d]$ must belong to $\mathcal{I}^{\preceq}$. For any $V\subset\mathcal{I}^{\preceq}$, we can give a total order. If $i=i_0\dots i_p, j=j_0\dots j_q\in V$ and $i\wedge j=i_0\dots i_{\ell-1}$, we denote $i<j$ if $i_\ell<j_\ell$. This means that for any $V\subset\mathcal{I}^{\preceq}$, there exists a bijection $\phi^V:V\to\{1,\dots,|V|\}$ associated with this total order in V.

Let $\hat{a}^m:[0,T]\times\left(\mathbb{R}^d\right)^m\to A^m$ be a function that is symmetric in the last m variables, for any $m \geq 1$. Let $\hat{\beta}$ be the control defined as follows

$$\hat{\beta}^i_s := \sum_{k\geq 1} \mathbb{1}_{\tau_{k-1}\leq s<\tau_k} \left(a_0 \mathbb{1}_{i\neq V_k} + \hat{a}^{|V_k|}_{\phi^{V_k}(i)} \left(s, Y_s^{(\phi^{V_k})^{-1}(1),\beta}, \dots, Y_s^{(\phi^{V_k})^{-1}(|V_k|),\beta}\right)\right). \quad (3.6.47)$$

Remark 3.6.8. *The connection between a control and a sequence of functions $\hat{a}^m$ provides insight into approaching the problem of optimal control through the examination of the corresponding HJB equation. The equation itself is dependent on v^m, where each branching event is associated with the switching of regime m.*

3.6.2 Verification Theorem

Theorem 3.6.13. *Let w be a function in $C^0\left([0,T]\times\mathcal{N}[\mathbb{R}^d]\right)$ such that*

$$-C_w\left(1+\langle 1,\lambda\rangle + \langle|\cdot|,\lambda\rangle\right) \leq w_t(\lambda) \leq C_w\left(1+\langle 1,\lambda\rangle^2 + \langle|\cdot|^2,\lambda\rangle\right). \quad (3.6.48)$$

for some constant $C_w > 0$. Assume that w^m, defined as in (3.6.46), is in $C^{1,2}\left([0,T]\times\mathbb{R}^{dm}\right)$ for any $m\in\mathbb{N}$.

(i) Suppose that

$$\begin{aligned} -\partial_t w^m(t,\vec{x}^m) - \inf_{\vec{a}^m\in A^m}\left\{\boldsymbol{L}^m w^m(\vec{x}^m,\vec{a}^m) + \sum_{i=1}^m \psi(x_i,\iota(\vec{x}^m),a_i)\right\} &\leq 0, \\ w^m(T,\vec{x}^m) &\leq \Psi(\iota(\vec{x}^m)), \end{aligned} \quad (3.6.49)$$

for any $m\in\mathbb{N}$, $t\in[0,T]$, and $\vec{x}^m\in\mathbb{R}^{dm}$. Then $w\leq v$ on $[0,T]\times\mathcal{N}[\mathbb{R}^d]$.

(ii) Suppose further $w^m(T,\vec{x}^m) = \Psi(\iota(\vec{x}^m))$, for any $m\in\mathbb{N}$, and $\vec{x}^m\in\mathbb{R}^{dm}$, and there exist measurable functions $\vec{\mathfrak{a}}^m(t,\vec{x}^m)$, for $m\in\mathbb{N}$, and $(t,\vec{x}^m)\in[0,T)\times\mathcal{N}[\mathbb{R}^d]$, valued in A^m such that

$$\begin{aligned} -\partial_t w^m(t,\vec{x}^m) &- \inf_{\vec{a}^m\in A^m}\left\{\boldsymbol{L}^m w^m(\vec{x}^m,\vec{a}^m) - \sum_{i=1}^m \psi(x_i,\iota(\vec{x}^m),a_i)\right\} \quad (3.6.50) \\ &= -\partial_t w^m(t,\vec{x}^m) - \left\{\boldsymbol{L}^m w^m(\vec{x}^m,\vec{\mathfrak{a}}^m(t,\vec{x}^m)) - \sum_{i=1}^m \psi(x_i,\iota(\vec{x}^m),\mathfrak{a}^m_i(t,\vec{x}^m))\right\} \\ &= 0. \end{aligned}$$

Defining $\hat{\beta}$ as in (3.6.47) associated with the functions $\vec{\mathfrak{a}}^m$ for $m\geq 1$, we assume that the following SDE admits a unique solution

$$\begin{aligned} \langle\varphi,\xi^{\hat{\beta}}_s\rangle &= \langle\varphi,\lambda\rangle + \int_t^s \sum_{i\in V_u} D\varphi(Y^{i,\hat{\beta}}_u)^\top\sigma\left(Y^{i,\hat{\beta}}_u,\xi^{\hat{\beta}}_u,\hat{\beta}^i_u\right)dB^i_u + \int_t^s \sum_{i\in V_u} L\varphi\left(Y^{i,\hat{\beta}}_u,\xi^{\hat{\beta}}_u,\hat{\beta}^i_u\right)du \\ &\quad + \int_{(t,s]\times\mathbb{R}_+} \sum_{i\in V_{u-}}\sum_{k\geq 0}(k-1)\varphi(Y^{i,\hat{\beta}}_u)\mathbb{1}_{I_k\left(Y^{i,\hat{\beta}}_u,\xi^{\hat{\beta}}_u,\hat{\beta}^i_u\right)}(z)Q^i(dudz). \end{aligned}$$

Suppose, moreover, that $\hat{\beta}\in\mathcal{R}^{\mathfrak{s}}_{(t,\lambda)}$ for any $(t,\lambda)\in\mathcal{N}[\mathbb{R}^d]$. Then, $w=v$ on $[0,T]\times\mathcal{N}[\mathbb{R}^d]$, and $\hat{\beta}$ is an optimal Markov control.

Proof. (i) We consider the notation adopted in Proposition 3.2.13. Fix a starting condition $(t, \vec{x}^m) \in [0,T] \times \mathbb{R}^{dm}$ and an admissible control $\beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\iota(\vec{x}^m))}$. Define the two sequences of stopping times $(\tau_k)_{k\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$

$$\begin{aligned}\tau_k &= \inf\left\{s \in (\tau_{k-1}, T] : \exists i \in V_{k-1},\ Q^i((\tau_{k-1}, s] \times [0, C_\gamma]) = 1\right\},\\ \theta_n &:= \inf\{s \in [t,T] : |V_s| \geq n\} \wedge \inf\left\{s \in [t,T] : \sum_{i\in V_u} \left|Y_u^{i,\beta}\right| \geq n\right\}.\end{aligned}$$

With these stopping times, we can describe ξ^β as

$$\xi^\beta_s = \sum_{k\geq 1} \mathbb{1}_{\tau_{k-1}\leq s<\tau_k} \sum_{i\in V_k} \delta_{Y_s^{i,\beta}} = \sum_{k\geq 1} \mathbb{1}_{\tau_{k-1}\leq s<\tau_k}\, \iota\left(\vec{Y}_s^{\beta,|V_k|}\right).$$

As noted in Remark 3.6.7, between the branching events τ_{k-1} and τ_k, the population behave like a controlled diffusion process living in $\mathbb{R}^{d|V_{k-1}|}$. Therefore, Itô's formula describes here the evolution of a function valued in ξ^β in each interval $[\tau_{k-1}\wedge\theta_n, \tau_k\wedge\theta_n)$.

Using the embedding ι, we have that (3.2.6) translates into

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{\mathfrak{m}_k^n}\left(s\wedge\tau_k\wedge\theta_n, \vec{Y}^{\beta,\mathfrak{m}_k^n}_{s\wedge\tau_k\wedge\theta_n}\right) - w^{\mathfrak{m}_{k-1}^n}\left(s\wedge\tau_{k-1}\wedge\theta_n, \vec{Y}^{\beta,\mathfrak{m}_{k-1}^n}_{s\wedge\tau_{k-1}\wedge\theta_n}\right)\right]\\ &\quad= \mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\int_{s\wedge\tau_{k-1}\wedge\theta_n}^{s\wedge\tau_1\wedge\theta_n} \left\{\partial_t w^{\mathfrak{m}_{k-1}^n}\left(t, \vec{Y}_u^{\beta,\mathfrak{m}_{k-1}^n}\right) + \mathbf{L}^{\mathfrak{m}_{k-1}^n} w^{\mathfrak{m}_{k-1}^n}\left(\vec{Y}_u^{\beta,\mathfrak{m}_{k-1}^n}, \vec{\beta}_u^{\mathfrak{m}_{k-1}^n}\right)\right\} du\right],\end{aligned}$$

where $\mathfrak{m}_k^n := |V_{\tau_k\wedge\theta_n}|$ and $\vec{\beta}_u^{\mathfrak{m}_{k-1}^n} := \left(\beta_u^i\right)_{i\in V_{\tau_{k-1}\wedge\theta_n}}$. Therefore, we have that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{|V_{s\wedge\theta_n}|}\left(s\wedge\theta_n, \vec{Y}^{\beta,|V_{s\wedge\theta_n}|}_{s\wedge\theta_n}\right)\right] - w^m(t, \vec{x}^m) \qquad (3.6.51)\\ &\quad= \mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\sum_{k\geq1}\left(w^{\mathfrak{m}_k^n}\left(s\wedge\tau_k\wedge\theta_n, \vec{Y}^{\beta,\mathfrak{m}_k^n}_{s\wedge\tau_k\wedge\theta_n}\right) - w^{\mathfrak{m}_{k-1}^n}\left(s\wedge\tau_{k-1}\wedge\theta_n, \vec{Y}^{\beta,\mathfrak{m}_{k-1}^n}_{s\wedge\tau_{k-1}\wedge\theta_n}\right)\right)\right]\\ &\quad= \mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\sum_{k\geq1}\int_{s\wedge\tau_{k-1}\wedge\theta_n}^{s\wedge\tau_1\wedge\theta_n} \left\{\partial_t w^{\mathfrak{m}_{k-1}^n}\left(t, \vec{Y}_u^{\beta,\mathfrak{m}_{k-1}^n}\right) + \mathbf{L}^{\mathfrak{m}_{k-1}^n} w^{\mathfrak{m}_{k-1}^n}\left(\vec{Y}_u^{\beta,\mathfrak{m}_{k-1}^n}, \vec{\beta}_u^{\mathfrak{m}_{k-1}^n}\right)\right\} du\right].\end{aligned}$$

Since w satisfies (3.6.50), we have

$$\partial_t w^{\mathfrak{m}_k^n}\left(t, \vec{Y}_u^{\beta,\mathfrak{m}_k^n}\right) + \mathbf{L}^{\mathfrak{m}_k^n} w^{\mathfrak{m}_k^n}\left(\vec{Y}_u^{\beta,\mathfrak{m}_k^n}, \vec{\beta}_u^{\mathfrak{m}_k^n}\right) + \sum_{i\in V_{\tau_k\wedge\theta_n}} \psi\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) \geq 0,$$

for any $\beta \in \mathcal{R}^{\mathfrak{s}}_{(t,\iota(\vec{x}^m))}$, $k \geq 0$ and $u \in [\tau_k\wedge\theta_n, \tau_{k+1}\wedge\theta_n)$. Thus,

$$\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{|V_{s\wedge\theta_n}|}\left(s\wedge\theta_n, \vec{Y}^{\beta,|V_{s\wedge\theta_n}|}_{s\wedge\theta_n}\right)\right] - w^m(t, \vec{x}^m) \geq -\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\int_t^{s\wedge\theta_n} \sum_{i\in V_u} \psi\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) du\right] \qquad (3.6.52)$$

From (3.2.15)-(3.2.16), we get

$$\left|\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right|\le C_\Psi\left(1+\int_t^T\left(|V_u|^2+\sum_{i\in V_u}\left|Y_u^{i,\beta}\right|^2+\sum_{i\in V_u}\left|\beta_u^i\right|^2\right)du\right),$$

therefore the r.h.s. in (3.6.52) is integrable for $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\iota(\vec{x}^m))}$ using (3.2.8), (3.2.19) and (3.2.20). Analogously, from (3.6.48), we also have that l.h.s. in (3.6.52) explodes to infinity or is integrable for $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\iota(\vec{x}^m))}$. We can then apply the dominated convergence theorem, and send n to infinity into (3.6.52), obtaining

$$\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{|V_s|}\left(s,\vec{Y}_s^{\beta,|V_s|}\right)\right]-w^m\left(t,\vec{x}^m\right)\ge-\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right],\ \text{for }\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\iota(\vec{x}^m))}.$$

Since w is continuous on $[0,T]\times\mathcal{N}[\mathbb{R}^d]$, by sending s to T, we obtain by the dominated convergence theorem and by (3.6.49) that

$$\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\Psi\left(\xi_T^\beta\right)\right]-w^m\left(t,\vec{x}^m\right)\ge-\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\int_t^T\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right],\ \text{for }\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\iota(\vec{x}^m))}.$$

From the arbitrariness of $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\iota(\vec{x}^m))}$, we deduce that $w^m\left(t,\vec{x}^m\right)\le v^m\left(t,\vec{x}^m\right)$, for any $m\ge1$, and $(t,\vec{x}^m)\in[0,T]\times\mathbb{R}^{dm}$, *i.e.*, $w\left(t,\lambda\right)\le v\left(t,\lambda\right)$ for any $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$.

(ii) From the definition of the control $\hat\beta$, we have that

$$-\partial_t w^m\left(t,\vec{x}^m\right)-\left\{\mathbf{L}^m v^m\left(\vec{x}^m,\vec{\mathfrak{a}}^m\left(t,\vec{x}^m\right)\right)-\sum_{i=1}^m\psi\left(x_i,\iota(\vec{x}^m),\mathfrak{a}_i^m\left(t,\vec{x}^m\right)\right)\right\}=0.$$

Applying this to (3.6.51), we get

$$w^m\left(t,\vec{x}^m\right)=\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{|V_{s\wedge\theta_n}|}\left(s\wedge\theta_n,\vec{Y}_{s\wedge\theta_n}^{\hat\beta,|V_{s\wedge\theta_n}|}\right)+\int_t^{s\wedge\theta_n}\sum_{i\in V_u}\psi\left(Y_u^{i,\hat\beta},\xi_u^{\hat\beta},\hat\beta_u^i\right)du\right],$$

for any $n\ge1$. From Fatou's lemma, we obtain

$$w^m\left(t,\vec{x}^m\right)\ge\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[w^{|V_s|}\left(s,\vec{Y}_s^{\hat\beta,|V_s|}\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\hat\beta},\xi_u^{\hat\beta},\hat\beta_u^i\right)du\right].$$

Sending s to T and using again Fatou's lemma, together with the fact $w^p\left(T,\vec{y}^p\right)=\Psi\left(\iota\left(\vec{y}^p\right)\right)$, for any $p\in\mathbb{N}$, and $\vec{y}^p\in\mathbb{R}^{dp}$, we see that

$$w^m\left(t,\vec{x}^m\right)\ge\mathbb{E}^{\mathbb{P}^{\mathfrak{s}}}\left[\Psi\left(\xi_T^{\hat\beta}\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\hat\beta},\xi_u^{\hat\beta},\hat\beta_u^i\right)du=J\left(t,\left(\vec{x}^m\right);\hat\beta\right).\right].$$

This shows that $w^m\left(t,\vec{x}^m\right)\ge J\left(t,\left(\vec{x}^m\right);\hat\beta\right)\ge v^m\left(t,\vec{x}^m\right)$, and finally that $w=v$ with $\hat\beta$ as an optimal Markovian control. □

The verification theorem presented here offers the advantage of not only establishing the optimality of a solution but also showing that a certain function is smaller than the value function. This characterization serves as a generalization of [152, Theorem II.3.1] for value functions in a broader context. However, this description differs significantly from the one used to introduce the controlled processes. Hence, to ensure the proof of optimality, we provide an equivalent verification theorem. The subsequent proposition establishes a characterization of optimality without relying on the embedding to $\cup_{m\geq 0}\mathbb{R}^{dm}$. Instead, it employs a (sub)martingale criterion similar to [137, Lemma 2.1].

Proposition 3.6.22. *Let w be a function in $C^0\left([0,T]\times\mathcal{N}[\mathbb{R}^d]\right)$ such that*

$$-C_w\left(1+\langle 1,\lambda\rangle+\langle|\cdot|,\lambda\rangle\right)\leq w_t(\lambda)\leq C_w\left(1+\langle 1,\lambda\rangle^2+\langle|\cdot|^2,\lambda\rangle\right). \tag{3.6.53}$$

for some constant $C_w>0$. Fix $(t,\bar\lambda)\in\mathcal{N}[\mathbb{R}^d]$, and assume the following

(i) $w_T(\lambda)=g(\lambda)$, for $\lambda\in\mathcal{N}[\mathbb{R}^d]$;

(ii) $\left\{w_s\left(\xi_s^\beta\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du : s\in[t,T]\right\}$ is a $\mathbb{P}^{\mathfrak{s}}$-local submartingale, for any $\beta\in\mathcal{R}^{\mathfrak{s}}_{(t,\bar\lambda)}$;

(iii) there exists $\hat\beta\in\mathcal{R}^{\mathfrak{s}}_{(t,\bar\lambda)}$ such that $\left\{w_s\left(\xi_s^{\bar\beta}\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\bar\beta},\xi_u^{\bar\beta},\bar\beta_u^i\right)du : s\in[t,T]\right\}$ is a $\mathbb{P}^{\mathfrak{s}}$-local martingale.

Then, $\bar\beta$ is an optimal control for $v(t,\bar\lambda)$, i.e., $v(t,\bar\lambda)=J(t,\bar\lambda;\bar\beta)$, and $v(t,\bar\lambda)=w_t(\bar\lambda)$.

Proof. By the local submartingale property in condition *(ii)*, there exists a nondecreasing sequence of stopping times $(\tau_n)_n$ such that $\tau_n\uparrow T$ a.s. and

$$\mathbb{E}\left[w_{s\wedge\tau_n}\left(\xi^\beta_{s\wedge\tau_n}\right)+\int_t^{s\wedge\tau_n}\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right]\geq w_t(\bar\lambda),\quad \text{for } \beta\in\mathcal{R}^{\mathfrak{s}}_{(t,\bar\lambda)}. \tag{3.6.54}$$

We fix $\varepsilon>0$ and restrict to consider $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\bar\lambda)}$. From (3.6.53) and (3.2.15)-(3.2.16), we see that for all n and $\beta\in\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\bar\lambda)}$, the r.h.s. is integrable and bounded by an integrable quantity. Applying dominated convergence theorem, by sending n to infinity into (3.6.54), we get

$$\begin{aligned}
w_t(\bar\lambda) &\leq \mathbb{E}\left[w_T\left(\xi_T^\beta\right)+\int_t^T\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right]\\
&\leq \mathbb{E}\left[g\left(\xi_T^\beta\right)+\int_t^T\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^\beta,\beta_u^i\right)du\right]=J(t,\bar\lambda;\beta),
\end{aligned}$$

using the terminal condition *(i)*, and (3.3.31). Since β is arbitrary in $\mathcal{R}^{\mathfrak{s},\varepsilon}_{(t,\bar\lambda)}$, this shows that $v(t,\bar\lambda)\geq w_t(\bar\lambda)$. To obtain the reverse inequality when the local martingale property for $\bar\beta$ in condition *(iii)* holds, we need to proceed as in the point *(iii)* of Theorem 3.6.13. This means that (3.6.54) is an equality and we conclude by applying Fatou's lemma. □

3.6.3 Examples

We include two examples within the linear-quadratic framework. By establishing the equivalence between weak controls and strong controls, we choose to utilize the former formalism due to its simpler notation, avoiding unnecessary complexities.

Standard Linear-Quadratic case

We follow the path outlined in [14, 137]. Let $A := \mathbb{R}^q$, $d' = d$ and let the coefficients be as follows

$$b_t(x,\lambda,a) = B_t x + \bar{B}_t a, \qquad \sigma_t(x,\lambda,a) = \sigma_t \mathbb{I},$$
$$\gamma_t(x,\lambda,a) = \gamma_t, \qquad p_k(x,\lambda,a) = p_k,$$

with $\mathbb{I}$ being the identity matrix, and B, $\bar{B}$, $\bar{\sigma}$, $\bar{\gamma}$ are bounded valued in $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times p}$, $\mathbb{R}^{d\times d}$ and $\mathbb{R}_+$ respectively. Since the control does not impact the coefficients that describe the branching, the search for a minimal control in (3.6.50) just focuses on each function w^m, without involving w^{m+k-1} for $k \geq 0$.

Let ψ and Ψ be as

$$\begin{aligned}\psi_t(x,\lambda,a) &= x^\top C_t x + c_t\langle 1,\lambda\rangle + a^\top \bar{C}_t a\\ \Psi(\lambda) &= \int_{\mathbb{R}^d} x^\top H x + h\langle 1,\lambda\rangle^2,\end{aligned}$$

where $t \mapsto C_t$ (resp. $t \mapsto \bar{C}_t$) is a bounded function in $\mathbb{S}^d$ (resp. $\mathbb{S}^m$), the set of symmetric matrices in $\mathbb{R}^{d\times d}$ (resp. $\mathbb{R}^{m\times m}$), $t \mapsto c_t \in \mathbb{R}_+$ is bounded, $H \in \mathbb{S}^d$ and $h \geq 0$.

We shall make the following assumptions

(i) C and H are non-negative a.s.;

(ii) $\bar{C}$ is uniformly positive definite, *i.e.*, $\bar{C}_t \geq \epsilon \mathbb{I}_m$ for some $\epsilon > 0$.

We are now ready to use Proposition (3.6.22) by seeking a field $\left\{w_t(\lambda) : \lambda \in \mathcal{N}[\mathbb{R}^d], t \in [0,T]\right\}$ that satisfies the local (sub)martingality conditions. Let w be as follows

$$w_t(\lambda) = w_t^1(\lambda) + w_t^2(\lambda) + w_t^3(\lambda), \qquad \text{with } w_t^1(\lambda) = \int_{\mathbb{R}^d} x^\top Q_t x \lambda(dx),$$
$$w_t^2(\lambda) = p_t\langle 1,\lambda\rangle^2, \quad w_t^3(\lambda) = \bar{p}_t\langle 1,\lambda\rangle,$$

for some funnctions $(Q,p,\bar{p})$ with values in $\mathbf{S}^d \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} dQ_t = \dot{Q}_t dt, & \text{for } t \in [0,T], \quad Q_T = H,\\ dp_t = \dot{p}_t dt, & \text{for } t \in [0,T], \quad p_T = h,\\ d\bar{p}_t = \dot{\bar{p}}_t dt, & \text{for } t \in [0,T], \quad \bar{p}_T = 0.\end{cases}$$

The terminal conditions ensure that $w_t(\lambda) = \Psi(\lambda)$. Now, we need to determine the generators $\dot{Q}$, $\dot{p}$ and $\dot{\bar{p}}$ to satisfy (3.6.50). Generalizing (3.3.22) to time-dependent functions, we have

$$\begin{aligned} w(t,\mu_t) \quad &+ \quad \int_0^t \int_{\mathbb{R}^d} \psi(x,\mu_u,\mathfrak{a}_u(x))\mu_u(dx)du\\ &= w(0,\mu_0) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_u(x,\mu_u,\mathfrak{a}_u(x),Q_u,p_u,\bar{p}_u)\mu_u(dx)du + \mathbb{M}_t, \end{aligned} \tag{3.6.55}$$

with

$$\begin{aligned}\mathcal{D}_u(x,\lambda,a,Q,p,\bar{p}) \quad := \quad & x^\top \dot{Q} x + \dot{p}\langle 1,\lambda\rangle + \dot{\bar{p}} + \left(B_u x + \bar{B}_u a\right)^\top Qx \\ & + x^\top Q\left(B_u x + \bar{B}_u a\right) + \sigma_u^2 \mathrm{Tr}(Q) + (\bar{\gamma}_u M_1) x^\top Q x \\ & + p\gamma_u \left(M_2 + M_1\langle 1,\lambda\rangle\right) + \bar{p}\gamma_u M_1 + x^\top C_u x + c_u\langle 1,\lambda\rangle + a^\top \bar{C}_u a\end{aligned}$$

and $\mathbb{M}$ is a martingale (after an eventual localization), and $M_1 := \sum_{k\geq 0}(k-1)p_k$, $M_2 := \sum_{k\geq 0}(k-1)^2 p_k$. Completing the square in $\mathcal{D}$, we obtain

$$\begin{aligned}\mathcal{D}_u(x,\lambda,a,Q,p,\bar{p}) := & \left(\dot{p} + p\gamma_u M_1 + c_u\right)\langle 1,\lambda\rangle + \left(\dot{\bar{p}} + \sigma_u^2 \mathrm{Tr}(Q) + \bar{p}\gamma_u M_1 + p\gamma_u M_2\right) \\ & + x^\top \left(\dot{Q} + B_u^\top Q + QB_u + (\bar{\gamma}_u M_1)Q + C_u + \left(\bar{B}_u Q + \bar{B}_u^\top Q\right)^\top \bar{C}_u^{-1}\left(\bar{B}_u Q + \bar{B}_u^\top Q\right)\right) x \\ & + (a - \hat{a}_u(x,Q))^\top \bar{C}_u (a - \hat{a}_u(x,Q)),\end{aligned}$$

where

$$\hat{a}_u(x,Q) := -\bar{C}_u^{-1}\left(\bar{B}_u Q + \bar{B}_u^\top Q\right) x.$$

Therefore, whenever

$$\begin{aligned}\dot{Q} + B_u^\top Q + QB_u + (\bar{\gamma}_u M_1)Q + C_u + 2Q\left(\bar{B}_u \bar{C}_u^{-1}\bar{B}_u + \bar{B}_u^\top \bar{C}_u^{-1}\bar{B}_u\right)Q &= 0, && (3.6.56)\\ \dot{p} + p\gamma_u M_1 + c_u &= 0, && (3.6.57)\\ \dot{\bar{p}} + \sigma_u^2 \mathrm{Tr}(Q) + \bar{p}\gamma_u M_1 + p\gamma_u M_2 &= 0, && (3.6.58)\end{aligned}$$

holds for $t \in [0,T]$, we have

$$\mathcal{D}_u(x,\lambda,a,Q,p,\bar{p}) = (a - \hat{a}_u(x,Q))^\top \bar{C}_u (a - \hat{a}_u(x,Q)).$$

Therefore, $\mathcal{D} \geq 0$ for any $a \in A$ and it is zero for $a = \hat{a}_u(x,Q)$. Additionally, it is worth noting that equations (3.6.56)-(3.6.58) have a solution since the first equation is a conventional Riccati equation, while the remaining two are linear ODEs.

This means that if the system of equations (3.6.58)-(3.6.58) is satisfied, from (3.6.55) and the fact that $\mathcal{D} \geq 0$, we get the local submartingale property *(ii)* of Proposition 3.6.22. Moreover, it is clear that it is zero for $\mathfrak{a}_u(x) := \hat{a}_u(x,Q)$, with Q solution to (3.6.56), satisfying the local martingale property *(iii)* of Proposition 3.6.22. Therefore, such a control is an optimal one.

A Kinetic Example

In the case of a standard diffusion process, we talk about *kinetic energy* when considering the following optimization setting. Consider controls β such that the diffusion process satisfies the following SDE

$$dX_t = \big(b(t,X_t) + \beta_s\big)dt + \sigma dB_t,$$

with b Lipschitz in x uniformly in t and σ a positive constant. We look for a minimization of the cost function $\mathbb{E}\left[\frac{1}{2}\int_0^T |\beta_s|^2\right]$, usually called the kinetic energy of the controlled diffusion process.

We adapt this framework to the case of branching processes. Let $A := \mathbb{R}^q$, $d' = d$ and consider the following

$$\begin{aligned} b_t(x,\lambda,a) &= b(t,x)+a, & \sigma_t(x,\lambda,a) &= \mathbb{I}, \\ \gamma_t(x,\lambda,a) &= \gamma_t(x), & p_k(x,\lambda,a) &= p_k(x), \end{aligned}$$

with b, γ and p_k satisfying (3.2.2), (3.2.3) and (3.2.4). Taking the running cost as $\psi(x,\lambda,a) := \frac{1}{2}|a|^2$, we seek for a field $\{w_t(\lambda) : \lambda \in \mathcal{N}[\mathbb{R}^d], t \in [0,T]\}$ of the following form

$$w_t(\lambda) = \int_{\mathbb{R}^d} h(t,x)\lambda(dx),$$

for a certain function h. From (3.3.22), we have

$$\begin{aligned} w\,(t,\mu_t) \quad + \quad & \int_0^t \int_{\mathbb{R}^d} \psi(x,\mu_u,\mathfrak{a}_u(x))\mu_u(dx)du \\ &= w\,(0,\mu_0) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_u(x,\mu_u,\mathfrak{a}_u(x),h)\mu_u(dx)du + \mathbb{M}_t, \end{aligned} \tag{3.6.59}$$

where

$$\mathcal{D}_t(x,\lambda,a,h) := \partial_t h + b(t,x)^\top Dh + a^\top Dh + \frac{1}{2}\Delta h + \frac{1}{2}|a|^2 + \phi(t,x)h,$$

with $\phi(x) := \gamma_t(x)\Big(\sum_{k\geq 0} kp_k(x) - 1\Big)$, $\mathbb{M}$ a martingale (after an eventual localization), and Δ the Laplacian. Operating as in the previous example, we see that whenever h satisfies the following PDE

$$\begin{cases} \partial_t h + b(t,x)^\top Dh - \frac{1}{2}|Dh|^2 + \frac{1}{2}\Delta h + \phi(t,x)h = 0 \\ h(T,x) = 0 \end{cases}, \tag{3.6.60}$$

we have

$$\mathcal{D}_u(x,\lambda,a,h) = \frac{1}{2}|a+Dh|^2.$$

This means that under (3.6.60), $\mathcal{D} \geq 0$ for any $a \in A$ and is zero for $a = -Dh$. Therefore, under (3.6.60), we get property *(ii)* of Proposition 3.6.22, and property *(iii)*, for $\mathfrak{a}_s(x) := -Dh(s,x)$, showing that this control is an optimal one. Under sufficient regularity of the function ϕ, the solution of (3.6.60) can be established and found as an application of the Hopf-Cole transformation.

3.7 Conclusion

Our study focused on proving the existence of an optimal solution for controlled branching diffusion processes with final and running costs. We presented the strong formalism, expanding it to cover controlled populations with linearly growing drifts. Furthermore, we established bounds that ensure proper problem definition, which strengthens and broadens the existing literature on the subject.

Given appropriate conditions, we introduced the concept of relaxed controls in this new setting. This differs from [44] on how we deal with the label of each particle and is more focused on the law of the process living in $M(\mathbb{R}^d)$. By defining natural and weak controls, we were able to narrow down the scope of the problem. Uniqueness was proved for the class of weak controls, with strong controls being associated with them. Through a Filippov-type convexity condition, we showed equivalence among all formulations. Shifting our focus to control rules, we deal with this class for its topological properties. We showed that the optimization problem can be confined to a compact set and that the cost function is lower semicontinuous. This guarantees the existence of an optimal value for the relaxed problem, and subsequently, for the strong problem as well.

An homeomorphism is established between $\mathcal{N}[\mathbb{R}^d]$ and $\cup_{m\geq 0}\mathbb{R}^{dm}$. Leveraging the differential properties of the latter space, we derive a system of HJB equations for the problem and establish a verification theorem by extracting a control from the minimization of the HJB equations. Finally, two linear-quadratic examples are presented with the use of these results.

Chapter 4

Controlled superprocesses and HJB equation in the space of finite measures

Outline of the current chapter

This chapter corresponds to the paper [127], which has been submitted for publication.

Abstract: This paper introduces the formalism required to analyze a certain class of stochastic control problems that involve a super diffusion process as the underlying controlled system. To establish the existence of these processes, we show that they are weak scaling limits of controlled branching processes. By proving their uniqueness in law, we can establish a dynamic programming principle for our stochastic control problem. This lays the groundwork for a PDE characterization of the associated value function, which is based on the concept of derivations in the space of finite positive measures. We also establish a verification theorem. To illustrate this approach, we

focus on an exponential-type value function and show how a regular solution to a finite-dimensional HJB equation can be used to construct a smooth solution to the HJB equation in the space of finite measures.

4.1 Introduction

The goal of population dynamics scaling limits is to provide a simpler method for modeling large populations. When particles can divide into more particles, we refer to them as branching particle systems. These systems are called branching diffusion processes when their movement through space follows a diffusion process.

Branching diffusion processes belong to a class of measure-valued processes that have received significant attention over the past thirty years. A good introduction to this topic can be found in [54]. The class of branching diffusion processes, as well as their scaling limits known as superprocesses, have been extensively studied in [72, 133, 141], among others. The purpose of this article is to introduce and examine the controlled counterpart of these processes, namely the controlled superprocesses.

Branching diffusion processes are discrete particle systems that underlie superprocesses, and there have been several studies on their controlled versions. This was first conducted in [152] and further developed in [125]. Along the same lines as the latter paper, [42] generalized this setting to controlled branching parameters and provided solutions to the problem using the *branching property* technique. This successful strategy involves emulating the symmetry of the problem to reduce it to a finite-dimensional optimization problem.

In particular, [42] and [125] focused on a cost of the same form. This is defined considering the product on all the particles alive at the terminal time of a continuous positive function taking values in the unit interval. Its associated value function can be seen as a cylindrical function of exponential type, which allows for the minimization of a global functional to be split into an optimization over the individual particles. Additionally, the fact that the coefficients are autonomous, meaning they only depend on the control and the position of each particle, translates the optimization into solving a finite-dimensional problem. This idea was used in Chapter 1 to address the stochastic target problem over branching particle systems. We will utilize this technique, in the final part of the article, to provide a class of control problems for which the solution can be explicitly computed.

The concept of breaking the coupling between individual actions and global population behavior has already appeared in the stochastic control literature. In particular, it is central to Mean Field Games (MFG) and Mean Field Control (MFC) problems. In this framework, the optimization problem for a large population is related to the control of a single participant interacting with the limit of the empirical measure of identical copies of itself. This is proven to converge to an interaction between a process and its law, satisfying a fixed point criterion. References for this topic can be found in [28, 31, 32, 117]. The existence of the controlled McKean–Vlasov dynamics is established as a weak limit of the law of interacting particle systems. In the MFC setting, for example, [33, 108, 107] use the relaxed control approach introduced in [64, 86] to show this limiting results. A similar weak reasoning method is used in [150], where the optimal stopping problem is generalized to the mean field setting using a control stopping strategy.

We aim to adopt a similar strategy for defining, weakly, the controlled limiting dynamics. Specifically, we utilize advancements in the analysis of controlled branching diffusion processes described in Chapter 3, which employs a relaxed setting that allows for a new characterization of these processes as weak controls. This weak control representation enables us to focus on the laws of these processes associated with starting condition, control, and martingale problem. By fixing the first two elements and manipulating the martingale problem, we prove that controlled superprocesses arise as a rescaling of branching processes, thereby establishing their existence. To achieve this, we extend the Aldous criterion presented in [54, 72, 133, 141] for convergence to superprocesses to a controlled setting. Furthermore, we generalize the martingale problem to a class of functionals that are convergence-determining in the space of càdlàg paths on finite measures and then use the ideas of [148], as detailed in [72] and Chapter 3, to establish uniqueness in law through the duality method.

Once existence and uniqueness have been shown, we focus on the related control problem and we adopt the Dynamic Programming Principle (DPP) approach. The DPP is a powerful tool for solving control problems and we achieve it by applying the methods described in [69, 70]. It has been shown (see, *e.g.*, [156]) that the DPP leads to a characterization of the problem through a nonlinear Hamilton–Jacobi–Bellman (HJB) equation. In our setting, the HJB needs to be defined as the space of finite measures. In the literature, PDEs on space of measures have already been investigated. MFG and MFC literature pushed the development of differential calculus in the space of probability measures to tackle this problem.

This approach leads to a verification theorem, which provides the necessary conditions for proving the optimality of a controlling strategy. This is achieved by showing that the value function of the control problem is a (viscosity) solution of this equation. This point of view has been explored in several works, including those focused on Markovian controls [138], open-loop controls [16], Markovian and non-Markovian frameworks [56], closed-loop controls [155], and McKean–Vlasov mixed regular-singular control problems [85]. An example of a study that combines the branching diffusion framework with the mean-field approach is presented in [44], where the authors introduce scaling limits that differ from the dynamics of superprocesses.

These techniques have then been extended to study dynamics on finite measures, as in [118]. In the latter article, the author studies backward Kolmogorov equations associated with stochastic filtering, entending the differential calculus developed for probability measures to general finite measures. Such an extension is presented introducing flat derivative and Lions' derivative with the same strategy of [30]. In the latter paper, the flat derivative is defined as a directional derivative, while the Lions' derivative is obtained as the derivation of the flat derivative with respect to the space component. This is done since the approach used by Lions in [117] cannot be employed in this new setting, as there is no lifting of the space of finite measures to that of L^2 random variables. Theoretical studies of the intrinsic properties of these differential operators can be found in [139], where intrinsic and extrinsic differentiations are introduced and shown to coincide with the notions of flat derivative and Lions' derivative in this context.

We adopt the differential calculus developed in [118], taking advantage of its density results to extend the martingale problem used to introduce these processes. This generalization is possible since the space of finite measures is homeomorphic to a subset of the product between probability measures and the real line when far from the measure zero. This allows, in particular, to use the lifting technique by renormalizing the measures whenever we are not close to this critical measure. This approach is similar to [47]. In this paper, the authors achieve the HJB equation and a verification theorem for the value functions using the density of the cylindrical functions in the space of continuously differentiable functions. We would like to emphasize that there are various other references available for tractable measure-valued processes, see, *e.g.*, [116, 149]. Exploring this direction could be valuable in establishing a basis for tractable control problems.

We will employ the generalized martingale problem on functions with sufficient differentiability relative to the measure. With this method, we provide the HJB equation and a verification theorem. The resulting Dynamic Programming Equation (DPE) features a second-order flat derivative. To the best of our knowledge, such a term has been seen before in a DPE only in [47], in the study of controlled probability measure-valued martingales. Nevertheless, this setting differs from ours where finite measures are involved. Moreover, the structure of the differential operators in our HJB equatons reveals a remarkable symmetry with the standard controlled diffusion processes, where a second-order operator appears in the PDE that describes the dynamics. Establishing the regularity of the value function poses a challenging problem in the context of solving non-linear PDEs, particularly for functions defined in measures. This question as well as the investigation of the viscosity solution to our DPE is left for future research.

The paper is organized as follows: In Section 4.2, we introduce the model setup as well as the controlled superprocesses as a solution to a martingale problem. We prove their uniqueness in law and existence as a weak limit of rescaled branching processes in Subsection 4.2.2. In particular, the latter is done with the use of the martingale problem for rescaled branching diffusion processes. We then show its convergences to a solution to the martingale problem defining the controlled superprocesses. We also establish a non-explosion bound, with respect to the metric metrizing weak* topology. In Section 4.3, we present the control problem of interest and prove its measurability property and the DPP. In Section 4.4, we derive the HJB equation satisfied by the value function of the control problem. To this purpose, we introduce a differential calculus in the space of finite measure and generalize the initial martingale problem using the density of cylindrical functions in the space of regular functions on finite measures. Finally, in Subsection 4.4.4, we derive the Dynamic Programming equation and prove a verification theorem. We conclude the paper by providing a regular solution to the optimization problem.

4.2 Controlled superprocesses

4.2.1 Model setup and definitions

For a Polish space (E, d) with $\mathcal{B}(E)$ its Borelian σ-field, we write $C_b(E)$ (resp. $C_0(E)$) for the subset of the continuous functions that are bounded (resp. that vanish at infinity), and $M(E)$ (resp. $\mathcal{P}(E)$) for the set of Borel positive finite measures (resp. probability measures) on E. We equip $M(E)$ with weak* topology, *i.e.*, the weakest topology that makes continuous the maps $M(E) \ni \lambda \mapsto \int_E \varphi(x)\lambda(dx)$ for any $\varphi \in C_b(\mathbb{R}^d)$. We denote $\langle \varphi, \lambda \rangle = \int_E \varphi(x)\lambda(dx)$ for $\lambda \in M(E)$ and $\varphi \in C_b(E)$.

A family $\mathscr{F} \subseteq C_b(E)$ is said to be *separating* if, whenever $\langle \varphi, \lambda \rangle = \langle \varphi, \lambda' \rangle$ for all $\varphi \in \mathscr{F}$, and some $\lambda, \lambda' \in M(E)$, we necessarily have $\lambda = \lambda'$. Since E is Polish, from the Portmanteau theorem (see, *e.g.*, [148, Theorem 1.1.1]), the set of uniformly continuous functions, for any metric equivalent to d, is separating. From Tychonoff's embedding theorem (see, *e.g.*, [154, Theorem 17.8]), $C_b(E)$ is also separable. Therefore, there exists a countable and separating family $\mathscr{F}_E = \{\varphi_k, k \in \mathbb{N}\}$ subset of $C_b(E)$ such that the function $E \ni x \mapsto 1$ belongs to $\mathscr{F}_E$ and $||\varphi_k||_\infty := \sup_E |\varphi_k| \leq 1$ for all $k \in \mathbb{N}$. We use this setting to define the following distance

$$\mathrm{d}_E(\lambda, \lambda') = \sum_{\varphi_k \in \mathscr{F}_E} \frac{1}{2^k} \left| \langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle \right|,$$

for $\lambda, \lambda' \in M(E)$. As in [148, Theorem 1.1.2], this distance d_E induces on $M(E)$ the weak* topology. Whenever $E = \mathbb{R}^d$, we adjust this metric to take into account useful differential properties. Let $\mathscr{F}_{\mathbb{R}^d}$ be taken as a subset of $C_b^2(\mathbb{R}^d)$, the set of real functions with bounded, continuous derivatives over $\mathbb{R}^d$ up to order two. We can take this set as separating since C^2 is dense in C^0 for local uniform convergence (application of [82, Theorem 8.14]). We define the distance

$$\mathrm{d}_{\mathbb{R}^d}(\lambda, \lambda') = \sum_{\varphi_k \in \mathscr{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} \left| \langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle \right|, \tag{4.2.1}$$

with $q_k = \max\{1, ||D\varphi_k||_\infty, ||D^2\varphi_k||_\infty\}$, and D and D^2 denote gradient and Hessian.

Atomic measures

We write $\mathcal{N}^n[E]$ for the space of atomic measures in E where each atom has a mass multiple of $1/n$, *i.e.*,

$$\mathcal{N}^n[E] := \left\{ \frac{k_i}{n} \sum_{i \in V} \delta_{x_i} \ : \ k_i \in \mathbb{N}, x_i \in E \text{ for } i \in V, \ V \subseteq \mathbb{N}, \ |V| < \infty \right\},$$

where $|V|$ is the cardinal of the set V. For $n \geq 1$, $\mathcal{N}^n[E]$ is a weakly* closed subset of $M(E)$. If E is a Polish space, *e.g.* a Euclidean space, $\cup_{n\in\mathbb{N}}\mathcal{N}^n[E]$ is dense in $M(E)$. First, the result is shown for probability measures. For the fundamental theorem of simulation (cf [130, Theorem 1.2]), there exists a Borel function $\varphi_\lambda : [0,1] \to E$, for any $\lambda \in \mathcal{P}(E)$, such that $\lambda = \mathrm{Leb}_{|[0,1]} \circ \varphi_\lambda^{-1}$, where $\mathrm{Leb}_{|[0,1]} \circ \varphi_\lambda^{-1}$ denotes the image measure by φ_λ^{-1} of the Lebesgue measure on the unit interval. With GlivenkoCantelli theorem (cf [130, Theorem 4.1]) we approximate the Lebesgue measure on the unit interval by probability measures $\lambda_n \in \mathcal{N}^n[E]$. We get the final result decomposing each finite measure λ as a probability measure times its total mass $\lambda([0,1])$ and using for the latter the approximation $\lfloor n\lambda(E)\rfloor/n$, where $\lfloor\cdot\rfloor$ denotes the integer part of a real number.

State space

Fix a finite time horizon $T > 0$. Let $\mathbf{D}^d = \mathbb{D}([0,T]; M(\mathbb{R}^d))$ be the set of càdlàg functions (right continuous with left limits) from $[0,T]$ to $M(\mathbb{R}^d)$. We endow this space with Skorohod metric $d_{\mathbf{D}^d}$ associated with the metric $\mathrm{d}_{\mathbb{R}^d}$, which makes it complete (see, *e.g.*, [20, Theorem 14.2]). For $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$, $\mathbb{P}_t \in \mathcal{P}(M(\mathbb{R}^d))$ denotes the time-t marginal of $\mathbb{P}$, *i.e.*, the image of $\mathbb{P}$ under the map $\mathbf{D}^d \ni \mu \mapsto \mu_t \in M(\mathbb{R}^d)$. Denote $\mathbf{D}^{n,d} = \mathbb{D}([0,T]; \mathcal{N}^n[\mathbb{R}^d])$, a closed subset of $\mathbf{D}^d$.

We consider the canonical space $\mathbf{D}^d$, with μ its canonical process, and $\mathbb{F}^\mu = \{\mathcal{F}_s^\mu\}_s$ the filtration generated by μ. Let a compact subset A of $\mathbb{R}^m$ representing the set of actions, and $\mathcal{A}$ the set of $\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\}_s$-predictable processes from $[0,T] \times \mathbb{R}^d$ to A. Finally, for a given $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ and a stopping time τ, we denote $\left(\mathbb{P}_\omega, \omega \in \mathbf{D}^d\right)$ a regular conditional probability distribution of $\mathbb{P}$ given $\mathcal{F}_\tau$ (see, *e.g.*, [148, Chapter 1.1]).

Definition

We consider the following assumptions. We are given dimensions $d, d' \in \mathbb{N}$, and the following bounded continuous functions

$$(b, \sigma, \gamma) : \mathbb{R}^d \times M(\mathbb{R}^d) \times A \quad \to \quad \mathbb{R}^d \times \mathbb{R}^{d \times d'} \times \mathbb{R}_+.$$

Suppose b and σ are Lipschitz uniformly in a, *i.e.*, there exist $L > 0$ such that

$$|b(x,\lambda,a) - b(x',\lambda',a)| + |\sigma(x,\lambda,a) - \sigma(x',\lambda',a)| \leq L\left(|x - x'| + \mathrm{d}_{\mathbb{R}^d}(\lambda,\lambda')\right),$$

for any $x, x' \in \mathbb{R}^d$, $\lambda, \lambda' \in M(\mathbb{R}^d)$, and $a \in A$.

In Chapter 3, various equivalent descriptions for branching particle systems are presented. Among these, we opt to use the formalism of weak controls as it involves less cumbersome notation. We adopt the same perspective to establish a definition for controlled superprocesses, which will subsequently facilitate the proof of their existence as a weak limit of the aforementioned branching processes.

Let L be the generator defined by

$$L\varphi(x,\lambda,a) \quad = \quad b(x,\lambda,a)^\top D\varphi(x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,\lambda,a)D^2\varphi(x)\right) \ ,$$

for $\varphi \in C_b^2(\mathbb{R}^d)$ Let also $\mathcal{L}$ be the generator defined by

$$\mathcal{L}F_\varphi(x,\lambda,a) \quad = \quad F'(\langle\varphi,\lambda\rangle)L\varphi(x,\lambda,a) + \frac{1}{2}F''(\langle\varphi,\lambda\rangle)\gamma(x,\lambda,a)\varphi^2(x),$$

where F_φ denotes the the cylindrical function $F_\varphi = F(\langle\varphi,\cdot\rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ For simplicity, we write $F'_\varphi(\lambda)$ for $F'(\langle\varphi,\lambda\rangle)$ and $F''_\varphi(\lambda)$ for $F''(\langle\varphi,\lambda\rangle)$.

We can now define the *controlled superprocess.*

Definition 4.2.12. *Fix* $(t,\lambda) \in [0,T]\times M\big(\mathbb{R}^d\big)$. *We say that* $(\mathbb{P},\alpha) \in \mathcal{P}(\boldsymbol{D}^d)\times\mathcal{A}$ *is a* Controlled superprocess*, and we denote* $(\mathbb{P},\alpha) \in \mathcal{R}_{(t,\lambda)}$*, if* $\mathbb{P}(\mu_t = \lambda) = 1$ *and the process*

$$M_s^{F_\varphi} = F_\varphi(\mu_t) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du \tag{4.2.2}$$

is a $(\mathbb{P},\mathbb{F})$*-martingale with quadratic variation*

$$\left[M^{F_\varphi}\right]_s = \int_t^s \left(F'_\varphi(\mu_u)\right)^2 \int_{\mathbb{R}^d} \gamma(x,\mu_u,\alpha_u(x))\varphi^2(x)\mu_u(dx)du \tag{4.2.3}$$

for any $F \in C_b^2(\mathbb{R})$*,* $\varphi \in C_b^2(\mathbb{R}^d)$*, and* $s \geq t$*.*

4.2.2 Existence and uniqueness

We first focus on the uniqueness in law for the controlled superprocesses. Using *Doob's functional representation theorem* (see, *e.g.*, [100, Lemma 1.13]), we remark that a $\left\{\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}_s\right\}_s$-predictable process α from $[0,T]\times\mathbb{R}^d$ to A boils down to be a predictable map $\mathfrak{a}$ such that

$$\mathfrak{a} : [0,T]\times\mathbb{R}^d\times\mathbf{D}^d \quad \to \quad A \tag{4.2.4}$$

$$\Big(s,x,(\mu_u)_{u\in[0,T]}\Big) \quad \mapsto \quad \mathfrak{a}\Big(s,x,(\mu_u)_{u\in[0,s]}\Big) = \alpha_s(x). \tag{4.2.5}$$

As in Section 3.4, we generalize the martingale problem (4.2.2) to a domain that characterizes the law of processes in $[0,T]\times\mathbf{D}^d$. To do so, we first introduce the domain of cylindrical functions $\mathcal{D} \subseteq C^0([0,T]\times\mathbf{D}^d)$ as the set of $F_{(f_1,\ldots,f_p)} : [0,T]\times\mathbf{D}^d \to \mathbb{R}$ of the form

$$F_{(f_1,\ldots,f_p)}(s,\mathbf{x}) = F\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\ldots,\langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right), \tag{4.2.6}$$

for $(s,\mathbf{x}) \in \mathbb{R}_+\times\mathbf{D}^d$ and some $p \geq 1$, $t_1,\ldots,t_p \in [0,T]$, $F \in C_b^2(\mathbb{R}^p)$, and $f_1,\ldots,f_p \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. For $f \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, we use the notation $Lf(s,x,\lambda,a) = Lf(s,\cdot)(x,\lambda,a)$. For a measurable function $\beta : [0,T]\times\mathbb{R}^d \to A$, we then define the operator $\mathbb{L}^\beta$ on $\mathcal{D}$ by

$$\begin{aligned}\mathbb{L}^\beta F_{(f_1,\dots,f_p)}(s,\mathbf{x}) &= DF\left(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\dots,\langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\right)^\top \mathfrak{L}^\beta\mathbf{f}(s,\mathbf{x})\\ &\quad+\frac{1}{2}\mathrm{Tr}\Big(\left\langle \mathfrak{S}^\beta\mathbf{f}(\mathfrak{S}^\beta\mathbf{f})^\top(s,\cdot),\mathbf{x}_s\right\rangle D^2F\big(\langle f_1(s\wedge t_1,\cdot),\mathbf{x}_{s\wedge t_1}\rangle,\dots,\\ &\qquad\qquad \langle f_p(s\wedge t_p,\cdot),\mathbf{x}_{s\wedge t_p}\rangle\big)\Big)\end{aligned}$$

with $t_0=0$, where

$$\begin{aligned}\mathfrak{L}^\beta\mathbf{f}(s,\mathbf{x}) &:= \begin{pmatrix} \mathbb{1}_{s\le t_1}\int_{\mathbb{R}^d}\partial_t f_1(s,x)+Lf_1(s,x,\mathbf{x}_s,\beta(s,x))\mathbf{x}_s(dx)\\ \vdots\\ \mathbb{1}_{s\le t_p}\int_{\mathbb{R}^d}\partial_t f_p(s,x)+Lf_p(s,x,\mathbf{x}_s,\beta(s,x))\mathbf{x}_s(dx)\end{pmatrix},\\ \mathfrak{S}^\beta\mathbf{f}(s,x,\mathbf{x}) &:= \begin{pmatrix}\mathbb{1}_{s\le t_1}f_1(s,x)\sqrt{\gamma(x,\mathbf{x}_s,\beta(s,x))}\\ \vdots\\ \mathbb{1}_{s\le t_p}f_p(s,x)\sqrt{\gamma(x,\mathbf{x}_s,\beta(s,x))}\end{pmatrix},\end{aligned}$$

for $(s,x,\mathbf{x})\in[0,T]\times\mathbb{R}^d\times\mathbf{D}^d$. Following the language of [74], we call the graph of $\mathcal{D}$ the *full* generator $\mathbb{G}$, with

$$\mathbb{G} := \{(g,\mathbb{L}^\cdot g) : g\in\mathcal{D}\}. \tag{4.2.7}$$

We define the domain $\mathcal{D}^T\subseteq C^0(M(\mathbb{R}^d))$ of the functions

$$F_{(f_1,\dots,f_p)}(\lambda) = F\left(\langle f_1,\lambda\rangle,\dots,\langle f_p,\lambda\rangle\right),\quad \lambda\in M(\mathbb{R}^d), \tag{4.2.8}$$

for some $p\ge 1$, $F\in C_b^2(\mathbb{R}^p)$, and $f_1,\dots,f_p\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. These functions are embedded in $\mathcal{D}$ when we consider functions as in (4.2.6) such that f_i does not depend on s and $t_i=T$, for $i=1,\dots,p$. Therefore, with abuse of notation, we say that $\mathbb{L}$ acts on $\mathcal{D}^T$ with the obvious adjustments.

Considering the canonical process $\mu\in\mathbf{D}^d$, we have that, if $(\mathbb{P},\alpha)\in\mathcal{R}_{(t,\lambda)}$ the process

$$\bar{M}_s^h := h(\mu_{s\wedge\cdot}) - \int_t^s \mathbb{L}^\alpha h(\mu_{u\wedge\cdot})du,\quad t\le u\le T, \tag{4.2.9}$$

is a $(\mathbb{P},\mathbb{F})$-martingale with quadratic variation equal to

$$\begin{aligned}\left[\bar{M}^h\right]_s &:= \int_t^s \mathrm{Tr}\Big(\Big\langle \mathfrak{S}^\alpha\mathbf{f}(\mathfrak{S}^\alpha\mathbf{f})^\top(s,\cdot),\mu_s\Big\rangle \\ &\qquad DF(DF)^\top\left(\langle f_1(s\wedge t_1,\cdot),\mu_{s\wedge t_1}\rangle,\dots,\langle f_p(s\wedge t_p,\cdot),\mu_{s\wedge t_p}\rangle\right)\Big)du,\end{aligned} \tag{4.2.10}$$

for any $h=F_{(f_1,\dots,f_p)}\in\mathcal{D}$. Therefore, we can finally prove uniqueness in law for the controlled superprocesses as follows.

Proposition 4.2.23. *Fix $(t,\lambda)\in[0,T]\times M(\mathbb{R}^d)$ and $\alpha\in\mathcal{A}$. There exists at most one $\mathbb{P}^{t,\lambda,\alpha}\in\mathcal{P}(\boldsymbol{D}^d)$ such that $(\mathbb{P}^{t,\lambda,\alpha},\alpha)\in\mathcal{R}_{(t,\lambda)}$.*

Proof. The proof is based on Proposition 3.4.17 and Theorem 3.4.11, whose proofs are the same as in our setting with respect to the operator $\mathbb{L}$. □

We can now consider the existence problem. The existence of solutions to martingale problems is usually proven as a weak limit of solutions to well-posed problems. Superprocesses, in particular, arise as scaling limits of branching particle systems (see, *e.g.*, [54, 72, 141]).

For $n \in \mathbb{N}$, let $\mathcal{L}^n$ be the generator defined on the cylindrical functions $F_\varphi = F(\langle \varphi, \cdot \rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, as

$$\begin{aligned}\mathcal{L}^n F_\varphi(x, \lambda, a) &= F'(\langle \varphi, \lambda \rangle) L\varphi(x, \lambda, a) + \frac{1}{2n} F''(\langle \varphi, \lambda \rangle) \left| D\varphi(x)\sigma(x, \lambda, a) \right|^2 \\ &\quad + \gamma(x, \lambda, a) n^2 \left(\frac{1}{2} F\left(\langle \varphi, \lambda \rangle - \frac{1}{n}\varphi(x) \right) \right. \\ &\quad \left. + \frac{1}{2} F\left(\langle \varphi, \lambda \rangle + \frac{1}{n}\varphi(x) \right) - F_\varphi(\lambda) \right). \end{aligned} \tag{4.2.11}$$

Definition 4.2.13. *Fix* $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. *We say that* $(\mathbb{P}, \alpha) \in \mathcal{P}(\boldsymbol{D}^d) \times \mathcal{A}$ *is a* n-rescaled branching diffusion process*, and we denote* $(\mathbb{P}, \alpha) \in \mathcal{R}^n_{(t,\lambda)}$*, if* $\mathbb{P}(\mu_t = \lambda_n) = 1$ *and the process*

$$M_s^{F_\varphi, n} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}^n F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du \tag{4.2.12}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale with quadratic variation*

$$\begin{aligned}\left[M^{F_\varphi, n} \right]_s &= \int_t^s \left(F'_\varphi(\mu_u) \right)^2 \int_{\mathbb{R}^d} \left(\frac{1}{n} \left| D\varphi(x)\sigma(x, \mu_u, \alpha_u(x)) \right|^2 + \right. \\ &\quad \left. \gamma(x, \mu_u, \alpha_u(x)) \varphi^2(x) \right) \mu_u(dx) du \end{aligned} \tag{4.2.13}$$

for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, *and* $s \geq t$.

Proposition 4.2.24. *Fix* $n \geq 1$ *and* $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. *For* $\alpha \in \mathcal{A}$*, there exists a* $\mathbb{P}^{t, \lambda_n, \alpha; n} \in \mathcal{P}(\boldsymbol{D}^d)$ *such that* $(\mathbb{P}^{t, \lambda_n, \alpha; n}, \alpha) \in \mathcal{R}^n_{(t, \lambda_n)}$.

Proof. Fix a $\alpha \in \mathcal{A}$. For $n = 1$, the existence of $\mathbb{P}^{t, \lambda_1, \alpha; 1} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda_1, \alpha; 1}, \alpha) \in \mathcal{R}^1_{(t, \lambda^1)}$, for any $(t, \lambda^1) \in [0, T] \times \mathcal{N}^1[\mathbb{R}^d]$, is discussed in Section 3.4.2. This is done for general horizons $T > 0$. Existence of $\mathbb{P}^{t, \lambda_n, \alpha; n} \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^{t, \lambda_n, \alpha; n}, \alpha) \in \mathcal{R}^n_{(t, \lambda_n)}$, for any $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$ stems from this. We denote $\bar{\mathcal{R}}^S_{(t, \lambda^1)}$ the set of 1-rescaled branching diffusion processes defined in the interval $[0, S]$, for $S > 0$. For $n \in \mathbb{N}$, we define, on the interval $[0, nT]$, the control α^n such that

$$\alpha^n_s = \alpha_{s/n}, \qquad \text{for } s \in [0, nT].$$

Fix $(t, \lambda_n) \in [0, T] \times \mathcal{N}^n[\mathbb{R}^d]$. From the previous result, we have the existence of $\mathbb{P}^{n,1} \in \mathcal{P}(\mathbb{D}([0, nT]; M(\mathbb{R}^d)))$ such that $(\mathbb{P}^{n,1}, \alpha^n) \in \bar{\mathcal{R}}^{nT}_{(nt, n\lambda_n)}$. Define the map $\mathfrak{R}^n$ such that

$$\begin{aligned}\mathfrak{R}^n : \mathbb{D}([0, T]; \mathcal{N}^n[\mathbb{R}^d]) &\to \mathbb{D}([0, nT]; \mathcal{N}^1[\mathbb{R}^d]) \\ (\mu_s)_{s \in [0, T]} &\mapsto (n\mu_{s/n})_{s \in [0, nT]}. \end{aligned}$$

As in [54, 72, 141], we have that $\mathbb{P}^{t, \lambda_n, \alpha; n} := \mathbb{P}^{n,1} \circ (\mathfrak{R}^n)^{-1} \in \mathcal{P}(\mathbf{D}^d)$ is such that $(\mathbb{P}^n, \alpha) \in \mathcal{R}^n_{(t, \lambda_n)}$. □

Now we have all the ingredients to give existence for controlled superprocesses.

Proposition 4.2.25. *Fix* $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$*. For* $\alpha \in \mathcal{A}$*, there exists a unique* $\mathbb{P} \in \mathcal{P}(\boldsymbol{D}^d)$ *such that* $(\mathbb{P},\alpha) \in \mathcal{R}_{(t,\lambda)}$*.*

Proof. Fix a $\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\}_s$-predictable process α from $[0,T] \times \mathbb{R}^d$ to A. Consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to \lambda$ weakly* and $\lambda_n \in \mathcal{N}^n[\mathbb{R}^d]$. From Proposition 4.2.24, there exists $\mathbb{P}^n \in \mathcal{P}(\mathbf{D}^d)$ such that $(\mathbb{P}^n,\alpha) \in \mathcal{R}^n_{(t,\lambda_n)}$. Our goal is to show $(\mathbb{P}^n)_{n\in\mathbb{N}}$ converges weakly to some $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$ and that $(\mathbb{P},\alpha) \in \mathcal{R}_{(t,\lambda)}$.

We define the projection π_φ as

$$\pi_\varphi : M(\mathbb{R}^d) \ni \lambda \mapsto \langle \varphi, \lambda\rangle \in \mathbb{R}$$

for any $f \in C^0(\mathbb{R}^d)$. Clearly, the weak* topology is the weakest topology for which the mappings π_{φ_k} are continuous, for $\{\varphi_k\}_{k\ge 1}$ dense in $C_0(\mathbb{R}^d)$. Moreover, under $\mathbb{P}^n$, for the semimartingale $\langle \varphi_k, \mu_\cdot\rangle$, we define the predictable finite variation process as $V^n_\cdot(\varphi)$ and the increasing process of the martingale part as $I^n_\cdot(\varphi)$ for $k \ge 1$. From equations (4.2.12) and (4.2.13), we have

$$\begin{aligned} V^n_s(\varphi_k) &= \int_t^s \int_{\mathbb{R}^d} L\varphi_k(x,\mu_u,\alpha_u(x))\mu_u(dx)du, && (4.2.14)\\ I^n_s(\varphi_k) &= \int_t^s \int_{\mathbb{R}^d} \bigg(\frac{1}{n}\left|D\varphi_k(x)\sigma(x,\mu_u,\alpha_u(x))\right|^2 && (4.2.15)\\ &\qquad + \gamma(x,\mu_u,\alpha_u(x))\varphi_k^2(x)\bigg)\mu_u(dx)du. \end{aligned}$$

We can now verify conditions *(i)* and *(ii)* of [141, Theorem 2.3] to prove that $\{\mathbb{P}^n\}_n$ is tight. This means proving that

(i) $\left(\mathbb{P}^n \circ \pi^{-1}_{\varphi_k}\right)_{n\ge 1}$ is tight for $k \ge 1$;

(ii) $V^n_s(\varphi_k)$ and $I^n_s(\varphi_k)$ satisfy the following condition of Aldous for any $k \ge 1$: for each stopping time τ we can find a sequence δ_n such that $\delta_n \to 0$ as $n \to \infty$ and such that

$$\begin{aligned} \limsup_n \mathbb{E}^{\mathbb{P}_n}\left[\left|V^n_{\tau+\delta_n}(\varphi) - V^n_\tau(\varphi)\right|\right] = 0,\\ \limsup_n \mathbb{E}^{\mathbb{P}_n}\left[\left|I^n_{\tau+\delta_n}(\varphi) - I^n_\tau(\varphi)\right|\right] = 0. \end{aligned} \qquad (4.2.16)$$

From (4.2.12), we have that $\langle 1,\mu_\cdot\rangle$ is a $\mathbb{P}^n$-martingale for any $n \ge 1$. Therefore, for any $n \ge 1$,

$$\mathbb{P}^n\left(\sup_{s\in[t,T]}\langle 1,\mu_s\rangle > K\right) \le \frac{1}{K}\mathbb{E}^{\mathbb{P}^n}\left[\langle 1,\mu_t\rangle\right] = \frac{1}{K}\langle 1,\lambda_n\rangle.$$

Since $\lim_n\langle 1,\lambda_n\rangle = \langle 1,\lambda\rangle$, we obtain that $\sup_n \mathbb{P}^n\left(\sup_{s\in[t,T]}\langle 1,\mu_s\rangle > K\right)$ tends to 0 when K tends to infinity. $\left(\mathbb{P}^n \circ \pi^{-1}_{\varphi_k}\right)_{n\ge 1}$ is also tight for $k \ge 1$, since each function of $C_0(\mathbb{R}^d)$ is bounded. Therefore, (i) is satisfied.

Fix $\varphi \in \{1\} \cup \{\varphi_k : k \ge 1\}$, a stopping time τ taking values in $[t,T]$, and $\delta_n > 0$. Using

(4.2.14) and (4.2.15), we have

$$\mathbb{E}^{\mathbb{P}_n}\left[\left|V^n_{\tau+\delta_n}(\varphi) - V^n_\tau(\varphi)\right|\right] \leq \mathbb{E}^{\mathbb{P}_n}\left[\int_\tau^{\tau+\delta_n}\int_{\mathbb{R}^d} |L\varphi(x,\mu_u,\alpha_u(x))|\mu_u(dx)du\right]$$
$$\leq \delta_n \mathbb{E}^{\mathbb{P}_n}\left[\langle 1,\mu_\tau\rangle\right]||L\varphi||_\infty = \delta_n\langle 1,\lambda_n\rangle||L\varphi||_\infty,$$

where the last equality comes from the martingale property. By the same arguments, we also have

$$\mathbb{E}^{\mathbb{P}_n}\left[\left|I^n_{\tau+\delta_n}(\varphi) - I^n_\tau(\varphi)\right|\right]$$
$$= \mathbb{E}^{\mathbb{P}_n}\left[\int_\tau^{\tau+\delta_n}\int_{\mathbb{R}^d}\left(\frac{1}{n}\left|D\varphi(x)\sigma(x,\mu_u,\alpha_u(x))\right|^2 + \gamma(x,\mu_u,\alpha_u(x))\varphi^2(x)\right)\mu_u(dx)du\right]$$
$$\leq \delta_n\langle 1,\lambda_n\rangle\left(||D\varphi\sigma||^2_\infty + 2\bar{\gamma}||\varphi||^2_\infty\right).$$

Therefore, if $\lim_n \delta_n = 0$, we get (4.2.16), which gives that $(\mathbb{P}_n)_{n\geq 1}$ is tight in $\mathbf{D}^d$.

To conclude, we take a sequence $(\mathbb{P}_n)_{n\geq 1}$ converging to a probability measure $\mathbb{P}\in\mathcal{P}(\mathbf{D}^d)$ and prove that $(\mathbb{P},\alpha)\in\mathcal{R}_{(t,\lambda)}$. To do that, we focus on the convergence of $\mathcal{L}_n$. For $(x,\nu,a)\in\mathbb{R}^d\times M(\mathbb{R}^d)$, the third term in the expression of $\mathcal{L}_n$ in (4.2.11) is equal to

$$W_n(x,\nu,a) = \gamma(x,\nu,a)n^2\left(F\left(\langle\varphi,\nu\rangle - \frac{1}{n}\varphi(x)\right)\frac{1}{2} + F\left(\langle\varphi,\nu\rangle + \frac{1}{n}\varphi(x)\right)\frac{1}{2} - F\left(\langle\varphi,\nu\rangle\right)\right).$$

Using Taylor's development with Lagrange reminder, we have

$$W_n(x,\nu,a) = \gamma(x,\nu,a)\frac{F''\left(\langle\varphi,\nu\rangle + z_1^n\right) + F''\left(\langle\varphi,\nu\rangle + z_2^n\right)}{2},$$

with z_1^n (resp. z_2^n) a point in $\{h\langle\varphi,\nu\rangle + (1-h)\varphi(x)/n : h\in[0,1]\}$ (resp. $\{h\langle\varphi,\nu\rangle - (1-h)\varphi(x)/n : h\in[0,1]\}$). Since γ is bounded, we have $W(x,\nu,a) = \lim_n W_n(x,\nu,a) = F''_\varphi(\nu)\gamma(x,\nu,a)\varphi^2(x)$ for any (x,ν,a). We can now prove that $F_\varphi(\mu_.) - \int_t^. \int_{\mathbb{R}^d}\mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du$ is a $\mathbb{P}$-martingale, *i.e.*, for each stopping time τ taking value in $[t,T]$,

$$\mathbb{E}^{\mathbb{P}}\left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau\int_{\mathbb{R}^d}\mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] = 0.$$

We have

$$\lim_n \mathbb{E}^{\mathbb{P}_n}\left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t)\right] = \mathbb{E}^{\mathbb{P}}\left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t)\right],$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] - \mathbb{E}^{\mathbb{P}_n}\left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}_n F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] - \mathbb{E}^{\mathbb{P}_n}\left[\int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] \\
&\qquad + \mathbb{E}^{\mathbb{P}_n}\left[\int_t^\tau \int_{\mathbb{R}^d} (\mathcal{L}-\mathcal{L}_n)F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right].
\end{aligned}$$

The last term on the right side satisfies

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{P}_n}\left[\int_t^\tau \int_{\mathbb{R}^d} (\mathcal{L}-\mathcal{L}_n)F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right]\right| \\
&= \left|\mathbb{E}^{\mathbb{P}_n}\left[\int_t^\tau \int_{\mathbb{R}^d} \left(\frac{1}{2n}F''(\langle\varphi,\mu_u\rangle)\,|D\varphi(x)\sigma(x,\mu_u,\alpha_u(x))|^2 + W(x,\mu_u,\alpha_u(x)) - W_n(x,\mu_u,\alpha_u(x))\right)\right.\right. \\
&\qquad \left.\left.\mu_u(dx)du\right]\right| \le \frac{C}{n}(1+T\langle 1,\lambda_n\rangle),
\end{aligned}$$

for a constant C which depends only on $F''_\varphi, \sigma, D\varphi, \gamma, \varphi$. Hence,

$$\begin{aligned}
&\lim_n \mathbb{E}^{\mathbb{P}_n}\left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}_n F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[F_\varphi(\mu_\tau) - F_\varphi(\mu_t) - \int_t^\tau \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right] = 0.
\end{aligned}$$

□

Remark 4.2.9. *It can be observed that the proof of the aforementioned result relies on either a compact control space or bounded coefficients. In the case of unbounded action space and linear dependence on the control in the coefficients, an integrability bound similar to the following must be established*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}^d} |\alpha_u(x)|\mu_u(dx)du\right] < \infty.$$

Applying this condition to the rescaled problem does not guarantee its retrieval in the limit.

4.2.3 Moment estimates

Before defining the control problem and proving it is well posed, we need to provide moment estimates for these processes. To do that, as in Chapter 3, we give the representation of the controlled superprocesses as Stochastic Differential Equations. This makes use of martingale measures, in extensions of the original space, and lets us apply the general theory of semimartingales in a more general setting. Relevant definitions and results on these objects are concisely summarised in [67] (see, *e.g.*, [153] for a monograph on the subject). We recall briefly their definition.

Definition 4.2.14. *Let $(G,\mathcal{G})$ be a Lusin space with its σ-algebra, and $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F}=\{\mathcal{F}_s\}_s)$ a filtered space satisfying the usual condition, where we define $\mathcal{P}$ the predictable σ-field. A process $\mathcal{M}$ on $\Omega\times[0,T]\times\mathcal{G}$ is called* martingale measure on G *if* *(i)* $\mathcal{M}_0(E) = 0$ *a.s. for any* $E \in \mathcal{G}$*;*

(ii) $\mathcal{M}_t$ *is a* σ*-finite,* $L^2(\Omega)$*-valued measure for all* $t \in [0,T]$*;*

(iii) $(\mathcal{M}_t(E))_{t\in[0,T]}$ *is an* $\mathbb{F}$*-martingale for any* $E \in \mathcal{G}$*.*

We say that $\mathcal{M}$ *is* orthogonal *if the product* $\mathcal{M}_t(E)\mathcal{M}_t(E')$ *is a martingale for any two disjoint sets* $E, E' \in \mathcal{G}$*. We also say, on one hand, that it is* continuous *if* $(\mathcal{M}_t(E))_{t\geq 0}$ *is continuous,* purely discontinuous*, on the other hand, if* $(\mathcal{M}_t(E))_{t\geq 0}$ *is a purely discontinuous martingale for any* $E \in \mathcal{G}$*.*

Proposition 4.2.26. *Let* $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$*. There exists an extension* $\Big(\hat{\Omega} = \boldsymbol{D}^d \times \tilde{\Omega}, \hat{\mathcal{F}} = \mathcal{F}^\mu_T \otimes \tilde{\mathcal{F}}, \hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}, \Big\{\hat{\mathcal{F}}_s = \mathcal{F}^\mu_s \otimes \tilde{\mathcal{F}}_s\Big\}_s\Big)$ *of* $\Big(\boldsymbol{D}^d, \mathcal{F}^\mu_T, \mathbb{P}, \mathbb{F}^\mu\Big)$*, where we naturally extend* μ *and* α*, that satisfies the following properties.*

1. $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ *is a filtered probability space supporting a continuous* $\hat{\mathbb{F}}$*-martingale measures* $\mathcal{M}$ *on* $\hat{\Omega} \times [0,T] \times \mathcal{B}(\mathbb{R}^d)$*, with intensity measure* $\mu_u(dx)du$*.*

2. $\hat{\mathbb{P}} \circ X_t^{-1} = \lambda$*.*

3. *We have that*

$$\begin{aligned}\langle f, \mu_s\rangle = \langle f, \mu_t\rangle &+ \int_t^s \int_{\mathbb{R}^d} Lf(x, \mu_u, \alpha_u(x))\mu_u(dx)du \\ &+ \int_t^s \int_{\mathbb{R}^d} \sqrt{\gamma(x, \mu_u, \alpha_u(x))} f(x)\mathcal{M}(dx, du)\ .\end{aligned} \tag{4.2.17}$$

for all $f \in C_b^\infty(\mathbb{R}^d)$ *and all* $s \in [t,T]$*.*

Proof. The representation of these processes is grounded in representation theorems for continuous martingale measures. We follow [121] and Proposition 3.3.15 applying their construction here. □

We can now prove the non-explosion of these processes, which will imply the well-posedness of the optimization problem.

Proposition 4.2.27. *Fix* $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$ *and* $p \in [1,2]$*. There exists a constant* $C \geq 0$*, depending only on* T*, and the coefficient of the parameters, such that*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,T]} d_{\mathbb{R}^d}(\mu_u, \mathbb{0})^p\right] \quad \leq \quad C d_{\mathbb{R}^d}(\lambda, \mathbb{0})^p, \tag{4.2.18}$$

for any $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$*.*

Proof. Fix $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}$. We recall that $\mathrm{d}_{\mathbb{R}^d}(\mu_u, \mathbb{0}) = \sum_{\varphi_k\in\mathscr{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} |\langle \varphi_k, \mu_u\rangle|$, for any $u \in [t,T]$. We define the stopping times τ_N as

$$\tau_N = \inf\left\{u \geq t : \langle 1, \mu_u\rangle \geq N\right\},$$

and denote $\mu^N_s := \mu_{\tau_N\wedge s}$, for $N \geq 1$. Proposition 4.2.26 implies that there exists an extension of Ω where μ can be satisfies (4.2.17) on the stochastic interval $[t, \tau_N]$. Such SDE is driven by

$\mathcal{M}^N$, a orthogonal continuous martingale measure in $[0,T]\times\mathbb{R}^d$, with the intensity measure $\mu_s(dx)\mathbb{1}_{s\le\tau_N}ds$. Applying (4.2.17) to φ_k, we have

$$\begin{aligned}\langle\varphi_k,\mu_s^N\rangle = \langle\varphi_k,\lambda\rangle \quad &+ \quad \int_t^s\int_{\mathbb{R}^d} L\varphi_k(x,\mu_r,\alpha_r(x))\mu_r(dx)\mathbb{1}_{r\le\tau_N}dr + \\ &+ \quad \int_t^s\int_{\mathbb{R}^d}\sqrt{\gamma(x,\mu_r,\alpha_r(x))}\varphi_k(x)\mathcal{M}^N(dx,dr).\end{aligned}$$

for $s\ge t$, and $k\in\mathbb{N}$. Applying Young's inequality, there is a constant C (which may change from line to line) such that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}|\langle\varphi_k,\mu_s^N\rangle|^p\right] \quad &\le \quad C|\langle\varphi_k,\lambda\rangle|^p \\ &\quad +C\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}\left|\int_t^s\int_{\mathbb{R}^d}L\varphi_k(x,\mu_r,\alpha_r(x))\mu_r(dx)\mathbb{1}_{r\le\tau_N}dr\right|^p\right] \\ &\quad +C\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}\left|\int_t^s\int_{\mathbb{R}^d}\sqrt{\gamma(x,\mu_r,\alpha_r(x))}\varphi_k(x)\mathcal{M}^N(dx,dr)\right|^p\right].\end{aligned}$$

Recalling $q_k=\max\{1,||D\varphi_k||_\infty,||D^2\varphi_k||_\infty\}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}|\langle\varphi_k,\mu_s^N\rangle|^p\right] \quad &\le \quad C|\langle\varphi_k,\lambda\rangle|^p + Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}\left(\int_t^s|\langle\varphi_k,\mu_u^N\rangle|du\right)^p\right] \\ &\quad +Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\int_t^T\langle 1,\mu_u^N\rangle^p du\right] \\ &\quad +C\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}\left|\int_t^s\int_{\mathbb{R}^d}\sqrt{\gamma(x,\mu_r,\alpha_r(x))}\varphi_k(x)\mathcal{M}^N(dx,dr)\right|^p\right].\end{aligned}$$

From Jensen's and Burkholder-Davis-Gundy's inequalities (see, *e.g.*, [55, Chapter VII, Theorem 92]), and recalling that $||\varphi_k||_\infty\le 1$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]}|\langle\varphi_k,\mu_s^N\rangle|^p\right] \quad &\le \quad C|\langle\varphi_k,\lambda\rangle|^p + Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\int_t^T|\langle\varphi_k,\mu_u^N\rangle|^p du\right] + Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\int_t^T\langle 1,\mu_u^N\rangle^p du\right] \\ &\le \quad Cq_k^p|\langle\varphi_k,\lambda\rangle|^p + Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\int_t^T\sup_{s\in[t,u]}|\langle\varphi_k,X_s^N\rangle|^p du\right] \\ &\qquad + Cq_k^p\mathbb{E}^{\mathbb{P}}\left[\int_t^T\sup_{s\in[t,u]}\langle 1,X_s^N\rangle^p du\right].\end{aligned}$$

Finally, multiplying by $\left(\frac{1}{2^k q_k}\right)^p$, summing over $k\in\mathbb{N}$ and applying the monotone convergence theorem, in addition to the fact that function equal to 1 is in $\mathscr{F}_{\mathbb{R}^d}$, we have

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,T]}\mathrm{d}_{\mathbb{R}^d}(X_u^N,\mathbb{0})^p\right] \quad \le \quad C\mathrm{d}_{\mathbb{R}^d}(\lambda,\mathbb{0}) + C\mathbb{E}^{\mathbb{P}}\left[\int_t^T\sup_{s\in[t,u]}\mathrm{d}_{\mathbb{R}^d}(X_s^N,\mathbb{0})^p du\right].$$

Using Grönwall's lemma, we conclude that $\mathbb{E}^{\mathbb{P}}\left[\sup_{u\in[t,T]} \mathrm{d}_{\mathbb{R}^d}(X_u^N,\mathbb{O})^p\right] \le C\mathrm{d}_{\mathbb{R}^d}(\lambda,\mathbb{O})^p$ for any $N\ge 1$. Applying Fatou's lemma, we obtain (4.2.18). □

4.3 The control problem

We are given two continuous functions $\psi:\mathbb{R}^d\times M(\mathbb{R}^d)\times A\to\mathbb{R}$ and $\Psi: M(\mathbb{R}^d)\to\mathbb{R}$. We assume that there exists $C>0$ such that

$$|\psi(x,\lambda,a)| \le C\left(1+\mathrm{d}_{\mathbb{R}^d}(\lambda,\mathbb{O})\right), \qquad |\Psi(\lambda)|\le C\left(1+\mathrm{d}_{\mathbb{R}^d}(\lambda,\mathbb{O})^2\right) \tag{4.3.19}$$

for $(x,\lambda,a)\in\mathbb{R}^d\times M(\mathbb{R}^d)\times A$ with $\mathbb{O}$ the measure 0.

Let J and v be respectively the cost and the value functions, defined as

$$J(t,\lambda,\alpha) = \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^T\int_{\mathbb{R}^d}\psi(x,\mu_s,\alpha_s(x))\mu_s(dx)ds+\Psi(\mu_T)\right], \tag{4.3.20}$$

$$v(t,\lambda) = \inf_{\alpha\in\mathcal{A}} J(t,\lambda,\alpha), \tag{4.3.21}$$

for $(t,\lambda)\in[0,T]\times M(\mathbb{R}^d)$, and $\alpha\in\mathcal{A}$. From Proposition 4.2.27, the cost function J is finite for any control $\alpha\in\mathcal{A}$. Moreover, using (4.3.19), J is uniformly bounded from below, therefore the optimization problem that defines v is well-posed.

4.3.1 Weak formulation

Before establishing the Dynamic Programming Principle (DPP), we give a new description of the control problem (4.3.20)-(4.3.21). As described in Section 3.5, we interpret the control set as a subset of finite measures. This is the so-called *weak formulation*, introduced in [65], and it allows for dealing with the control space and its topology more flexibly.

Consider $[0,T]\times\mathbb{R}^d\times A$ equipped with the σ-algebra $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(A)$. Let $\mathfrak{A}^{\mathrm{Leb}}\subseteq M([0,T]\times\mathbb{R}^d\times A)$ be the set of measures, whose projection on $[0,T]$ is the Lebesgue measure. Each $\alpha\in\mathfrak{A}^{\mathrm{Leb}}$ can be identified with its disintegration (see, *e.g.*, [101, Corollary 1.26, Chapter 1]). In particular, we have $\bar\alpha(ds,dx,da) = ds\mathbf{y}_s(dx)\bar\alpha_s(x,da)$, for a process $(\mathbf{y}_s(dx))_s$ (resp. $(\bar\alpha_s(x,da))_s$) taking values in the set of functions from $[0,T]$ (resp. $[0,T]\times\mathbb{R}^d$) to $M(\mathbb{R}^d)$ (resp. $M(A)$).

We denote $\bar\Omega := \mathbf{D}^d\times\mathfrak{A}^{\mathrm{Leb}}$. On $\bar\Omega$, let (μ,β) be the projection maps (or canonical processes), and $\mathbb{F}^{\mu,\beta}=\left\{\mathcal{F}_s^{\mu,\beta}\right\}_s$ the filtration generated by these maps, *i.e.*,

$$\mathcal{F}_s^{\mu,\beta} = \sigma\left(\mu_s(B),\beta([0,r]\times B'\times C),\ \text{for } s,r\in[0,T], B,B'\in\mathcal{B}(\mathbb{R}^d), C\in\mathcal{B}(A)\right).$$

Moreover, define the following map

$$\begin{aligned}\pi_{\mathcal{A}}:\mathbf{D}^d\times\mathcal{A} &\to \mathfrak{A}^{\mathrm{Leb}}\\ (\mathbf{x},\alpha) &\mapsto ds\mathbf{x}_s(dx)\delta_{\alpha_s(x)}(da).\end{aligned}$$

Definition 4.3.15. *Fix* $(t,\lambda)\in[0,T]\times\mathcal{N}[\mathbb{R}^d]$. *We say that* $\mathbb{P}\in\mathcal{P}(\bar\Omega)$ *is a* weak control rule, *and we denote* $\mathbb{P}\in\mathcal{C}_{(t,\lambda)}$, *if* $\mathbb{P}(\mu_t=\lambda)=1$, *there exists* $\alpha^{\mathbb{P}}\in\mathcal{A}$ *such that* $\mathbb{P}\left(\pi_{\mathcal{A}}\left(\mu,\alpha^{\mathbb{P}}\right)=\beta\right)=1$, *and the process*

$$M_s^{F_\varphi} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d\times A} \mathcal{L}F_\varphi(x,\mu_u,a)\beta_s(x,da)\mu_u(dx)du$$

is a $\left(\mathbb{P},\mathbb{F}^{\mu,\beta}\right)$*-martingale with quadratic variation*

$$\left[M^{F_\varphi}\right]_s = \int_t^s \left(F'_\varphi(\mu_u)\right)^2 \int_{\mathbb{R}^d\times A} \gamma(x,\mu_u,a)\beta_s(x,da)\varphi^2(x)\mu_u(dx)du$$

for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, *and* $s \geq t$.

It is clear that each element of $\mathcal{C}_{(t,\lambda)}$ can be identified to an element of $\mathcal{R}_{(t,\lambda)}$, and viceversa. With abuse of notation, we write $J(t,\lambda,\mathbb{P})$ for $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$ to denote $J(t,\lambda,\alpha^{\mathbb{P}})$. With this description, we have

$$v(t,\lambda) = \inf_{\alpha\in\mathcal{A}} J(t,\lambda,\alpha) = \inf_{\mathbb{P}\in\mathcal{C}_{(t,\lambda)}} J(t,\lambda,\mathbb{P}).$$

In this framework, we can consider the notion of conditioning as well as concatenation on $\bar{\Omega}$. For $(t,\bar{w}) \in [0,T]\times\bar{\Omega}$, we denote

$$\begin{aligned} \mathfrak{P}_{\bar{w}}^t &:= \{\bar{\omega} : \mu_t(\bar{\omega}) = \mu_t(\bar{w})\}, \\ \mathfrak{P}_{t,\bar{w}} &:= \left\{\bar{\omega} : (\mu_s, \mathcal{M}_s(\phi))(\bar{\omega}) = (\mu_s, \mathcal{M}_s(\phi))(\bar{w}),\ \text{for } s\in[0,t], \phi \in C_b\left([0,T]\times\mathbb{R}^d\times A\right)\right\}, \end{aligned}$$

where

$$\mathcal{M}_s(\phi) := \int_0^s \int_{\mathbb{R}^d\times A} \phi(s,x,a)\beta(ds,dx,da).$$

Then, for all $\bar{\omega} \in \mathfrak{P}_{\bar{w}}^t$, we define the concatenated path $\bar{w}\otimes_t\bar{\omega}$ by

$$(\mu_s,\mathcal{M}_s(\phi))(\bar{w}\otimes_t\bar{\omega}) = \begin{cases} (\mu_s,\mathcal{M}_s(\phi))(\bar{w}), & \text{for } s\in[0,t), \\ (\mu_s,\mathcal{M}_s(\phi)-\mathcal{M}_t(\phi))(\bar{\omega}) + (\mu_s,\mathcal{M}_t(\phi))(\bar{w}), & \text{for } s\in[t,T], \end{cases}$$

for all $\phi \in C_b\left([0,T]\times\mathbb{R}^d\times A\right)$.

Fix $\mathbb{P}\in\mathcal{P}(\bar{\Omega})$, and τ a $\mathbb{F}^{\mu,\beta}$-stopping time. From [156, Proposition 1.9, Chapter 1], there is a family of regular conditional probability distribution (r.c.p.d.) $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega}\in\bar{\Omega}}$ w.r.t. $\mathcal{F}_\tau^{\mu,\beta}$ such that the $\mathcal{F}_\tau^{\mu,\beta}$-measurable probability kernel $(\mathbb{P}_{\bar{\omega}})_{\bar{\omega}\in\bar{\Omega}}$ satisfies

$$\mathbb{P}_{\bar{\omega}}\left(\mathfrak{P}_{\tau(\bar{\omega}),\bar{\omega}}\right) = 1 \quad \text{for } \mathbb{P}-\text{a.e. } \bar{\omega}\in\bar{\Omega}.$$

On the other hand, take a probability measure $\mathbb{P}$ defined on $\left(\bar{\Omega},\mathcal{F}_\tau^{\mu,\beta}\right)$ and a family of probability measures $(\mathbb{Q}_{\bar{\omega}})_{\bar{\omega}\in\bar{\Omega}}$ such that $\bar{\omega}\mapsto\mathbb{Q}_{\bar{\omega}}$ is $\mathcal{F}_\tau^{\mu,\beta}$-measurable and

$$\mathbb{Q}_{\bar{\omega}}\left(\mathfrak{P}_{\bar{\omega}}^{\tau(\bar{\omega})}\right) = 1 \quad \text{for } \mathbb{P}-\text{a.e. } \bar{\omega}\in\bar{\Omega}.$$

There is a unique concatenated probability measure that we denote $\mathbb{P}\otimes_\tau\mathbb{Q}.$ defined by

$$\mathbb{P}\otimes_\tau\mathbb{Q}.(C) \quad := \quad \int_{\bar{\Omega}} \mathbb{P}(d\bar{w}) \int_{\bar{\Omega}} \mathbb{1}_C\left(\bar{w}\otimes_{\tau(\bar{w})}\bar{\omega}\right)\mathbb{Q}_{\bar{w}}(d\bar{\omega}) \quad \text{for } C\in\mathcal{F}_T^{\mu,\beta}.$$

4.3.2 Measurable selection and DPP

This weak formulation has the advantage of simplifying the proof of the DPP. We follow the path detailed in [69] and [70], which clarify [19, Chapter 7] in the context of stochastic control theory, generalizing it to our setting. In particular, to reach the DPP, as in [69, Theorem 4.10] and [70, Theorem 3.1], we need to show that the setting so far presented satisfies [70, Assumption 2.2], which in our setting reads as follows.

It is clear that the *full* generator $\mathbb{G}$ as defined in (4.2.7) is countably generated. Combining Proposition 4.2.25 with the weak formulation description, $\mathcal{C}_{(t,\lambda)}$ is nonempty for all $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$. Fix $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$, $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$ and τ a $\mathbb{F}^{\mu,\beta}$-stopping time taking value in $[t,T]$. Using [70, Lemma 3.2] and [70, Lemma 3.3], we obtained

- *Stability by conditioning*: There is a family of r.c.p.d. $(\mathbb{P}_{\bar\omega})_{\bar\omega\in\bar\Omega}$ w.r.t. $\mathcal{F}^{\mu,\beta}_\tau$ such that $\mathbb{P}_{\bar\omega} \in \mathcal{C}_{(\tau(\bar\omega),\mu(\bar\omega))}$ for $\mathbb{P}$-a.e. $\bar\omega \in \bar\Omega$.
- *Stability by concatenation*: Let $(\mathbb{Q}_{\bar\omega})_{\bar\omega\in\bar\Omega}$ be a probability kernel from $\mathcal{F}^{\mu,\beta}_\tau$ into $\left(\bar\Omega, \mathcal{F}^{\mu,\beta}_T\right)$ such that $\bar\omega \mapsto \mathbb{Q}_{\bar\omega}$ is $\mathcal{F}^{\mu,\beta}_\tau$-measurable, and $\mathbb{Q}_{\bar\omega} \in \mathcal{C}_{(\tau(\bar\omega),\mu(\bar\omega))}$ for $\mathbb{P}$-a.e. $\bar\omega \in \bar\Omega$. Then, $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{C}_{(t,\lambda)}$.

These two conditions are those of [70, Assumption 2.2]. This allows us to prove the following DPP.

Theorem 4.3.14. *We have*

$$\begin{aligned} v(t,\lambda) &= \inf_{\mathbb{P}\in\mathcal{C}_{(t,\lambda)}} \mathbb{E}^{\mathbb{P}}\left[\int_t^\tau \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + v(\tau,\mu_\tau)\right] \\ &= \inf_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^\tau \int_{\mathbb{R}^d} \psi(x,\mu_s,\alpha_s(x))\mu_s(dx)ds + v(\tau,\mu_\tau)\right], \end{aligned} \tag{4.3.22}$$

for any $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$, and τ stopping time taking value in $[t,T]$.

Proof. Fix $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$, and τ to be a stopping time taking values in $[t,T]$. We have that the cost function (4.3.20) is continuous, thus a fortiori upper semi-analytic. Following the stability by conditioning, for any $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$, there is $(\mathbb{P}_{\bar\omega})_{\bar\omega\in\bar\Omega}$ a family of r.c.p.d. w.r.t. $\mathcal{F}^{\mu,\beta}_\tau$ such that $\mathbb{P}_{\bar\omega} \in \mathcal{C}_{(\tau(\bar\omega),\mu(\bar\omega))}$ for $\mathbb{P}$-a.e. $\bar\omega \in \bar\Omega$. Therefore, we get

$$J\left(\tau(\bar\omega),\mu_{\tau(\bar\omega)}(\bar\omega),\mathbb{P}_{\bar\omega}\right) = \mathbb{E}^{\mathbb{P}_{\bar\omega}}\left[\int_\tau^T \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + \Psi(\mu_T)\right], \quad \text{for } \mathbb{P}-\text{a.e. } \bar\omega \in \bar\Omega.$$

Since, by definition, $v(\tau(\bar\omega),\mu_{\tau(\bar\omega)}(\bar\omega)) \le J\left(\tau(\bar\omega),\mu_{\tau(\bar\omega)}(\bar\omega),\mathbb{P}_{\bar\omega}\right)$, it follows from the tower property of conditional expectations that

$$\begin{aligned} J(t,\lambda,\mathbb{P}) &= \int_{\bar\Omega}\left(J(\tau(\bar\omega),\mu_{\tau(\bar\omega)}(\bar\omega),\mathbb{P}_{\bar\omega}) + \int_t^\tau \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds\right)\mathbb{P}(d\bar\omega) \\ &\ge \mathbb{E}^{\mathbb{P}}\left[v(\tau,\mu_\tau) + \int_t^\tau \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds\right]. \end{aligned}$$

which provides $v(t,\lambda) \ge \inf_{\mathbb{P}\in\mathcal{C}_{(t,\lambda)}} \mathbb{E}^{\mathbb{P}}\left[\int_t^\tau \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + v(\tau,\mu_\tau)\right]$ by the arbitrariness of $\mathbb{P}$.

We now turn to the reverse inequality. Fix some arbitrary $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$ and $\varepsilon > 0$. Consider the set $\mathcal{C}^{\varepsilon}_{(t',\lambda')}$ defined as follows

$$\mathcal{C}^{\varepsilon}_{(t',\lambda')} \quad := \quad \left\{\mathbb{Q} \in \mathcal{C}_{(t',\lambda')} : v(t',\lambda') + \varepsilon \geq J(t',\lambda',\mathbb{Q})\right\}, \qquad \text{for } (t',\lambda') \in [0,T] \times M\big(\mathbb{R}^d\big).$$

From the [69, Proposition 2.21], there exists a family of probability $(\mathbb{Q}^{\varepsilon}_{\bar\omega})_{\bar\omega\in\bar\Omega}$ from $\mathcal{F}^{\mu,\beta}_{\tau}$ into $\left(\bar\Omega, \mathcal{F}^{\mu,\beta}_{T}\right)$ such that $\bar\omega \mapsto \mathbb{Q}^{\varepsilon}_{\bar\omega}$ is $\mathcal{F}^{\mu,\beta}_{\tau}$-measurable, and $\mathbb{Q}^{\varepsilon}_{\bar\omega} \in \mathcal{C}^{\varepsilon}_{(\tau(\bar\omega),\mu(\bar\omega))}$ for $\mathbb{P}$-a.e. $\bar\omega \in \bar\Omega$. Then, $\mathbb{P} \otimes_{\tau} \mathbb{Q}^{\varepsilon}_{\cdot} \in \mathcal{C}_{(t,\lambda)}$ by the stability by concatenation condition. This implies that

$$\begin{aligned}
J(t,\lambda,\mathbb{P}\otimes_{\tau}\mathbb{Q}^{\varepsilon}_{\cdot}) \quad &= \quad \mathbb{E}^{\mathbb{P}\otimes_{\tau}\mathbb{Q}^{\varepsilon}_{\cdot}}\left[\int_t^T \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + \Psi(\mu_T)\right] \\
&= \quad \int \Bigg(\int_t^{\tau(\bar\omega)} \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds \\
&\qquad\qquad + \mathbb{E}^{\mathbb{Q}^{\varepsilon}_{\bar\omega}}\left[\int_{\tau}^T \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + \Psi(\mu_T)\right]\Bigg)\mathbb{P}(d\bar\omega) \\
&= \quad \mathbb{E}^{\mathbb{P}}\left[\int_t^{\tau(\bar\omega)} \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + J\left(\tau(\bar\omega),\mu_{\tau(\bar\omega)}(\bar\omega),\mathbb{Q}^{\varepsilon}_{\bar\omega}\right)\right] \\
&\leq \quad \mathbb{E}^{\mathbb{P}}\left[\int_t^{\tau} \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + v\left(\tau,\mu_{\tau}\right)\right] + \varepsilon.
\end{aligned}$$

From the arbitrariness of $\mathbb{P} \in \mathcal{C}_{(t,\lambda)}$ and $\varepsilon > 0$, we obtain the inequality

$$v(t,\lambda) \leq \inf_{\mathbb{P}\in\mathcal{C}_{(t,\lambda)}} \mathbb{E}^{\mathbb{P}}\left[\int_t^{\tau} \int_{\mathbb{R}^d\times A} \psi(x,\mu_s,a)\beta_s(x,da)\mu_s(dx)ds + v(\tau,\mu_{\tau})\right].$$

□

4.4 Dynamic Programming Equation

The DPP opens the way to the characterization of the value function as a (viscosity) solution to nonlinear PDE. This approach links the (optimal) controlled dynamics of the processes under analysis in a "weak" way. This means that we are interested in the behavior of $s \mapsto u(\mu_s)$ for a certain class of test functions u.

For this purpose, first, we need to analyze the differential property of the space of finite measure. There exists a growing literature about differential calculus in the space of probability measures. This is due to the development of the mean field games theory. The two main objects discussed in this context are the linear functional derivative (also called flat derivative or extrinsic derivative) and the L-derivative (also called intrinsic derivative). The first is defined directly in $\mathcal{P}(\mathbb{R}^d)$, while the second relies on the lifting on a Hilbert space. It is found that one is the spatial gradient of the previous one, coinciding with the notion of derivative of [117]. Therefore, sometimes this is the definition used for the L-derivatives, like in [29]. Detailed discussions of this topic can be found for example in [30, 31, 29].

Readjusting these concepts to $M\big(\mathbb{R}^d\big)$, we present the same two notions, as introduced also in [118]. A survey on how these notions of derivatives intertwine is [139], where their properties are studied in a more general setting.

4.4.1 Differential properties

Definition 4.4.16 (Linear derivative)**.** *A function $u : M(\mathbb{R}^d) \to \mathbb{R}$ is said to have* linear derivative *if it is continuous, bounded and if there exists a function*

$$\delta_\lambda u : M(\mathbb{R}^d) \times \mathbb{R}^d \ni (\lambda, x) \mapsto \delta_\lambda u(\lambda, x) \in \mathbb{R},$$

that is bounded, and continuous for the product topology, such that

$$u(\lambda) - u(\lambda') = \int_0^1 \int_{\mathbb{R}^d} \delta_\lambda u\left(t\lambda + (1-t)\lambda', x\right)(\lambda - \lambda')(dx)\,dt,$$

for $\lambda, \lambda' \in M(\mathbb{R}^d)$. We call $C^1(M(\mathbb{R}^d))$ the class of functions from $M(\mathbb{R}^d)$ to $\mathbb{R}$ that are differentiable in linear functional sense.

Notice that $\delta_\lambda u$ is uniquely defined up to a constant. We take

$$\int_{\mathbb{R}^d} \delta_\lambda u(\lambda, x)\lambda(dx) = 0$$

as a convention in this paper. Moreover, second-order derivatives are introduced for $u \in C^1(M(\mathbb{R}^d))$ imposing that $\lambda \mapsto \delta_\lambda u(\lambda, x)$ is differentiable in linear functional sense for every x and that $(\lambda, x, y) \mapsto \delta^2_\lambda u(\lambda, x, y)$ is bounded and continuous. We call $C^2(M(\mathbb{R}^d))$ this class of functions.

Finite positive measures could not rely on lifting procedures. For this reason, the notion of *intrinsic* derivative is introduced deriving with respect to the x component the flat derivative, as done in [30, Definition 2.2].

Definition 4.4.17 (Intrinsic derivative)**.** *Fix $u \in C^1(M(\mathbb{R}^d))$. If $\delta_\lambda u$ is of class C^1 with respect to the second variable, the intrinsic derivative $D_\lambda u : M(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is*

$$D_\lambda u(\lambda, x) = \partial_x \delta_\lambda u(\lambda, x).$$

We denote with $C^{1,1}(M(\mathbb{R}^d))$ this class of functions.

Deriving with respect to the measure or the space component are two different operations. We denote $C^{k,\ell}(M(\mathbb{R}^d))$ with $k \in N$ to be the collection of functions u that are differentiable k times with respect to the measure and such that the k-th derivative with respect to the measure is ℓ-th times continuously differentiable with respect to its spatial components.

Remark 4.4.10. *As in [118, Example 2.9], we have that $\mathcal{D}^T \subseteq C^{2,2}(M(\mathbb{R}^d))$, where $\mathcal{D}^T$ is the domain of cylindrical functions as in* (4.2.8)*. In particular, it holds that*

$$\begin{aligned}
\delta_\lambda h(\lambda, x) &= DF\left(\langle f_1, \lambda\rangle, \ldots, \langle f_p, \lambda\rangle\right)^\top \mathbf{f}(x),\\
\delta^2_\lambda h(\lambda, x, y) &= \mathbf{f}(y)^\top DF^2\left(\langle f_1, \lambda\rangle, \ldots, \langle f_p, \lambda\rangle\right)\mathbf{f}(x),\\
D_\lambda h(\lambda, x) &= DF\left(\langle f_1, \lambda\rangle, \ldots, \langle f_p, \lambda\rangle\right)^\top D\mathbf{f}(x),
\end{aligned}$$

with $\mathbf{f}(x) := (f_1, \ldots, f_p)(x)^\top$, and $D\mathbf{f}(x) = (Df_1, \ldots, Df_p)(x)^\top$, for $h = F_{(f_1,\ldots,f_p)} \in \mathcal{D}^T$.

4.4.2 Density properties

To get the Dynamic Programming Equation associated with the value function v we first need to generalize the martingale problem (4.2.2). As in [85] and [118], to do this we first restrict ourselves in a compact space, prove the density of $\mathcal{D}^T$ in $C^{2,2}(M(\mathbb{R}^d))$ and conclude with a localization argument.

First, we restrict the space we work into a compact set. This is done to apply the Stone-Weierstrass theorem and prove the density of cylindrical functions. Consider the set of compact rectangles $\{K_N := [-N,N]^d\}_{N\geq 1} \subseteq \mathbb{R}^d$. For every $k, N \in \mathbb{N}$, we define the

$$\begin{aligned} M_k(\mathbb{R}^d) &:= \left\{\lambda \in M(\mathbb{R}^d) : \lambda(\mathbb{R}^d) \in \left[\frac{1}{k}, k\right]\right\}, \\ \mathcal{K}_N^k &:= \left\{\lambda \in M_k(\mathbb{R}^d) : \operatorname{supp}(\lambda) \subseteq K_N\right\}. \end{aligned}$$

These sets are non-empty. In particular, $\mathcal{K}_N^k$ is compact for the weak* topology for any $k, N \in \mathbb{N}$. Indeed, this space is homeomorphic to $\mathcal{K}_N^{k,1} \times \left[\frac{1}{k}, k\right]$ with $\mathcal{K}_N^{k,1} := \mathcal{K}_N^k \cap \mathcal{P}(\mathbb{R}^d)$ with the homeomorphism

$$\begin{aligned} \mathfrak{H} : M(\mathbb{R}^d)\backslash\{\mathbb{O}\} &\to \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}_+ \\ \lambda &\mapsto \left(\frac{1}{\lambda(\mathbb{R}^d)}\lambda, \lambda(\mathbb{R}^d)\right). \end{aligned}$$

The set $\mathcal{K}_N^{k,1}$ is weakly* precompact, using Prokhorov's theorem (see *e.g.* [20, Theorem 1.6.1]), and is closed as any limit point of the sequence in $\mathcal{K}_N^{k,1}$ also has support contained in K_N. Therefore, $\mathcal{K}_N^k$ is compact as homeomorphic to the product of two compact sets.

Given $\lambda \in M_k(\mathbb{R}^d)$, we denote with $\rho^N\mu$ the measure such that $\frac{d\rho^N\lambda}{d\lambda} = \rho^N$, with ρ^N a positive and smooth cut-off function equal to 1 in K_N and identically zero outside K_{N+1}. We observe that $\rho^N\lambda$ is always in $\mathcal{K}_{N+1}^k$. Thus, we set

$$u^N(\lambda) := u(\rho^N\lambda), \qquad \text{for } \lambda \in M_k(\mathbb{R}^d), N \geq 1. \tag{4.4.23}$$

With these notations, we give [118, Lemma 3.3] that proves the first approximation theorem for functions on $M_k(\mathbb{R}^d)$.

Lemma 4.4.8. *Fix $k \geq 1$ and $u \in C^{2,2}(M_k(\mathbb{R}^d))$. Let $\{u^N\}_{N\geq 1}$ be the sequence defined by (4.4.23). Then, for every $\lambda \in M_k(\mathbb{R}^d)$, $u^N(\lambda) \to u(\lambda)$ as $n \to \infty$ and $\{\delta_\lambda u^N\}_{N\geq 1}$, $\{\delta_\lambda^2 u^N\}_{N\geq 1}$, $\{D_\lambda u^N\}_{N\geq 1}$, and $\{\partial_x D_\lambda u^N\}_{N\geq 1}$ pointwise converge to the respective derivatives of u. Moreover, $||u^N||_\infty \leq ||u||_\infty$, and there exists $C > 0$ independent of u, N, and k such that*

$$\begin{aligned} ||\delta_\lambda u^N||_\infty &\leq ||\delta_\lambda u||_\infty, \\ ||\delta_\lambda^2 u^N||_\infty &\leq ||\delta_\lambda^2 u||_\infty, \\ ||D_\lambda u^N||_\infty &\leq C\left(||D_\lambda u||_\infty + ||\delta_\lambda u||_\infty\right), \\ ||\partial_x D_\lambda u^N||_\infty &\leq C\left(||D_\lambda u||_\infty + ||\delta_\lambda u||_\infty + ||\partial_x D_\lambda u||_\infty\right). \end{aligned}$$

This result allows us to approximate functions in $C^{2,2}(M_k(\mathbb{R}^d))$ with functions in $C^{2,2}(\mathcal{K}_N^k)$, for $k, N \in \mathbb{N}$. We can now adapt [118, Lemma 3.4] showing that the domain of cylindrical functions $\mathcal{D}^T$ is dense in $C^{2,2}(\mathcal{K}_N^k)$.

Lemma 4.4.9. *Fix $k \geq 1$ and $N \geq 1$. Let u be in $C^{2,2}(\mathcal{K}_N^k)$. There exists a sequence of cylindrical functions $\{u_n\}_{n\geq 1} \subseteq \mathcal{D}^T$ such that $u_n(\lambda) \to u(\lambda)$ as $n \to \infty$ and $\{\delta_\lambda u_n\}_{n\geq 1}$, $\{\delta_\lambda^2 u_n\}_{n\geq 1}$, $\{D_\lambda u_n\}_{n\geq 1}$, and $\{\partial_x D_\lambda u_n\}_{n\geq 1}$ converge pointwise to the respective derivatives of u for any $\lambda \in \mathcal{K}_N^k$. Moreover, $||u_n||_\infty \leq ||u||_\infty$, and the same holds for the derivatives, up to a multiplica-*

tive constant independent of u, N and k.

Since $\mathcal{K}_N^k$ is compact in $M\big(\mathbb{R}^d\big)$, the previous lemma proves

$$||u_n - u||_{C_b^{2,2}(M(\mathbb{R}^d))} \to 0, \quad \text{as } n \to \infty,$$

where

$$||u||_{C_b^{2,2}(M(\mathbb{R}^d))} := \sup_{\lambda \in \mathcal{K}_N^k, x, y \in K_N} \Big\{ |u(\lambda)| + |\delta_\lambda u(\lambda, x)| + |\delta_\lambda^2 u(\lambda, x, y)| + |D_\lambda u(\lambda, x)| + |\partial_x D_\lambda u(\lambda, x)| \Big\}, \tag{4.4.24}$$

for u in $C^{2,2}(M\big(\mathbb{R}^d\big))$ which is bounded with bounded derivatives. A stronger norm could be used by adding in the supremum also the terms depending on $\partial_x \delta_\lambda^2 u$ and $D_\lambda^2 u$ as in [118]. For our scope, norm (4.4.24) is enough to generalize the martingale problem (4.2.2).

Remark 4.4.11. *A different approach could have been taken to prove that $\mathcal{D}^T$ is dense in $C^{2,2}(\mathcal{K}_N^k)$. As proven in [85], this domain separates points in $\mathcal{K}_N^k$ and vanishes at no point, therefore using Stone-Weierstrass theorem, it is dense in the $C^0(\mathcal{K}_N^k)$ with the topology of the strong convergence. Then, using the definition for the Linear derivative and the intrinsic derivative, the convergences of the different derivatives could be established, as in [85, Lemma 3.12].*

4.4.3 Generalized martingale problem

We define the operator $\mathbf{L}$ on $u \in C_b^{2,2}(M\big(\mathbb{R}^d\big))$ by

$$\begin{aligned} \mathbf{L}u(\lambda, x, a) &= b(x, \lambda, a)^\top D_\lambda u(\lambda, x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x, \lambda, a)\partial_x D_\lambda u(\lambda, x)\right) \\ &\quad + \frac{1}{2}\gamma(x, \lambda, a)\delta_\lambda^2 u(\mu, x, x) \end{aligned}$$

for $(x, \lambda, a) \in \mathbb{R}^d \times M\big(\mathbb{R}^d\big) \times A$.

Proposition 4.4.28. *For $(t, \lambda) \in [0, T] \times M\big(\mathbb{R}^d\big)$ and $\alpha \in \mathcal{A}$, the following are equivalent:*

(i) $\big(\mathbb{P}^{t,\lambda,\alpha}, \alpha\big) \in \mathcal{R}_{(t,\lambda)}$*;*

(ii) the process

$$M_s^u = u(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \boldsymbol{L}u(x, \mu_u, \alpha_u(x))\mu_u(dx)du \tag{4.4.25}$$

is a $(\mathbb{P}, \mathbb{F})$-martingale with quadratic variation

$$[M^u]_s = \int_t^s \gamma(x, \mu_u, \alpha_u(x)) \left|\delta_\lambda u(\mu_u, x)\right|^2 \mu_u(dx)du \tag{4.4.26}$$

for any $u \in C_b^{2,2}(\mathbb{R}^d)$, and $s \geq t$.

Proof. $(ii) \implies (i)$: From Remark 4.4.10, $F_\varphi \in C_b^{2,2}(\mathbb{R}^d)$ for any $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ and equations (4.4.25) and (4.4.26) becomes (4.2.2) and (4.2.3) for $u = F_\varphi$.

$(i) \implies (ii)$: If $\lambda = \mathbb{O}$, it is clear that M^u is constant in time, thus a martingale with a null quadratic variation. Consider a starting condition (t, λ), with $\langle 1, \lambda \rangle > 0$, a control $\alpha \in \mathcal{A}$ and the sequence of stopping times $\{\tau_k\}_{k \geq 1}$ as

$$\tau_k := \inf \{s \geq t : \langle 1, \mu_s \rangle > k\} \wedge \inf \left\{s \geq t : \langle 1, \mu_s \rangle < \frac{1}{k}\right\}.$$

Defining $\mu^k_\cdot := \mu_{\cdot \wedge \tau_k}$, for $\langle 1, \lambda \rangle \in [1/k, k]$, we have that under $\mathbb{P}^{(t,\lambda,\alpha)}$, this process lives in $M_k(\mathbb{R}^d)$ by construction. Thus, applying Lemma 4.4.8 and Lemma 4.4.9, there exists a sequence $u_n \in \mathcal{D}^T$ such that $u_n \to u$ as $n \to \infty$ pointwise as well as their derivatives and there exists $C > 0$ such that $||u_n||_{C_b^{2,2}(M(\mathbb{R}^d))} \leq C||u||_{C_b^{2,2}(M(\mathbb{R}^d))}$.

Remark 4.4.10 shows how derivatives operate on cylindrical functions. Looking at (4.2.9) and (4.2.10), we see that for $h \in \mathcal{D}^T$ equations (4.4.25) and (4.4.26) are satisfied if applied to μ^k. We prove now that $u(\mu^k_\cdot) - \int_t^\cdot \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu_u, \alpha_u(x))\mu_u(dx)du$ is a $\mathbb{P}^{(t,\lambda,\alpha)}$-martingale, *i.e.*,, for each stopping time θ in $[t, T]$,

$$\mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[u(\mu^k_\theta) - u(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u(x, \mu^k_u, \alpha_u(x))\mu^k_u(dx)du\right] = 0.$$

Since (4.4.25) is satisfied for $h \in \mathcal{D}^T$, and from the bounds on the derivatives and on the coefficients b, σ and γ, we can apply the Dominated Convergence Theorem and obtain

$$\begin{aligned} 0 \;&=\; \lim_n \mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[u_n\left(\mu^k_\theta\right) - u_n(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u_n\left(x, \mu^k_u, \alpha_u(x)\right)\mu^k_u(dx)du\right] \\ &= \mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[u\left(\mu^k_\theta\right) - u(\lambda) - \int_t^\theta \int_{\mathbb{R}^d} \mathbf{L}u\left(x, \mu^k_u, \alpha_u(x)\right)\mu^k_u(dx)du\right]. \end{aligned}$$

By definition of the quadratic variation and (4.4.26) applied to $u_n \in \mathcal{D}^T$, we have for $n \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[\left(u_n\left(\mu^k_s\right) - u_n(\lambda) - \int_t^s \int_{\mathbb{R}^d} \mathbf{L}u_n\left(x, \mu^k_u, \alpha_u(x)\right)\mu^k_u(dx)du\right)^2\right] \\ &\qquad = \mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[\int_t^s \gamma\left(x, \mu^k_u, \alpha_u(x)\right)\left|\delta_\lambda u_n\left(\mu^k_u, x\right)\right|^2 \mu^k_u(dx)du\right] \end{aligned} \tag{4.4.27}$$

Therefore, we apply again Dominated Convergence Theorem again and obtain (4.4.27) with respect to u.

Finally, we can remove the localization using the Dominated Convergence Theorem since $u \in C_b^{2,2}(\mathbb{R}^d)$ and the bound (4.2.18). □

4.4.4 HJB Equation

We are ready to introduce the HJB equation associated with this control problem. Looking at (4.4.25), define an operator H on $\mathbb{R}^d \times M(\mathbb{R}^d) \times A \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}$ such that

$$\begin{aligned} H(x, \lambda, a, p, M, r) = b(x, \lambda, a)^\top p &+ \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x, \lambda, a)M\right) \\ &+ \frac{1}{2}\gamma(x, \lambda, a)r + \psi(x, \lambda, a). \end{aligned} \tag{4.4.28}$$

Then, if the value function (4.3.21) is sufficiently smooth, generalizing Proposition 4.4.28 to function depending in time and measure yields the following HJB equation

$$\begin{cases} \partial_t v(t,\lambda) + \int_{\mathbb{R}^d} \inf_{a\in A} H\Big(x,\lambda,a,D_\lambda v(t,\lambda,x), \\ \qquad\qquad\qquad \partial_x D_\lambda v(t,\lambda,x), \delta_\lambda^2 v(t,\lambda,x,x)\Big)\lambda(dx) = 0 \\ v(T,\lambda) = \Psi(\lambda) \end{cases}$$

The form of this HJB equation looks like the one for mean field control (see, *e.g.*, [16, 56, 85, 138, 155]), where the infimum (or the supremum if maximizing) is taken inside the integral. The major differences here are that we consider the space of finite measures and not only probability measures and that the second-order flat derivative is explicitly involved in the Hamiltonian.

We have the following result in the regular case.

Theorem 4.4.15 (Verification Theorem)**.** *Let $V : [0,T] \times M(\mathbb{R}^d) \to \mathbb{R}$ be a function living in $C_b^{1,(2,2)}([0,T) \times M(\mathbb{R}^d)) \cap C^0([0,T] \times M(\mathbb{R}^d))$.*

(i) Suppose that V satisfies

$$\begin{cases} \partial_t V(t,\lambda) + \int_{\mathbb{R}^d} \inf_{a\in A} H\Big(x,\lambda,a,D_\lambda V(t,\lambda,x), \\ \qquad\qquad\qquad \partial_x D_\lambda V(t,\lambda,x), \delta_\lambda^2 V(t,\lambda,x,x)\Big)\lambda(dx) \le 0 \\ V(T,\lambda) \le \Psi(\lambda). \end{cases} \tag{4.4.29}$$

Then $V(t,\lambda) \le v(t,\lambda)$ for any $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$, with v as in (4.3.21).

(ii) Moreover, suppose that V satisfies (4.4.29) with equality and there exists a continuous function $\hat{a}(t,x,\lambda)$ for $(t,x,\lambda) \in [0,T) \times \mathbb{R}^d \times M(\mathbb{R}^d)$, valued in A such that

$$\hat{a}(t,x,\lambda) \in \arg\min_{a\in A} H\left(x,\lambda,a,D_\lambda v(t,\lambda,x), \partial_x D_\lambda v(t,\lambda,x), \delta_\lambda^2 v(t,\lambda,x,x)\right). \tag{4.4.30}$$

Suppose also that the corresponding control $\alpha^* = \{\alpha_s^*(x) := \hat{a}(s,x,\mu_s), s \in [t,T)\} \in \mathcal{A}$. Then $V = v$, with v as in (4.3.21), and α^* is an optimal Markovian control.

Proof. (i) Since $V \in C_b^{1,(2,2)}([0,T) \times M(\mathbb{R}^d))$, we have for all $(t,\lambda) \in [0,T) \times M(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$, by (4.4.25), the process

$$V(s,\mu_s) - V(t,\lambda) - \int_t^s \partial_t V(u,\mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u,x,\mu_u,\alpha_u(x))\mu_u(dx)du$$

is a martingale under $\mathbb{P}^{t,\lambda,\alpha}$. By taking the expectation, we get

$$\mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[V(s,\mu_s)\right] = V(t,\lambda) + \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^s \partial_t V(u,\mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u,x,\mu_u,\alpha_u(x))\mu_u(dx)du\right]$$

Since w satisfies (4.4.29), we have

$$\partial_t V(u,\mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u,x,\mu_u,\alpha_u(x)) + \psi(x,\mu_u,\alpha_u(x))\mu_u(dx) \le 0, \quad \mathbb{P}^{t,\lambda,\alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. Therefore,

$$\mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[V(s,\mu_s)\right] \leq V(t,\lambda) - \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^s \int_{\mathbb{R}^d} \psi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right], \quad \mathbb{P}^{t,\lambda,\alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. Since V is continuous on $[0,T]\times M(\mathbb{R}^d)$, we obtain by the dominated convergence theorem and by (4.4.29)

$$\mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\Psi(\mu_T)\right] \leq V(t,\lambda) - \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^T \int_{\mathbb{R}^d} \psi(x,\mu_u,\alpha_u(x))\mu_u(dx)du\right], \quad \mathbb{P}^{t,\lambda,\alpha} - \text{a.s.}$$

for any $\alpha \in \mathcal{A}$. From the arbitrariness of the control, we deduce that $V(t,\lambda) \leq v(t,\lambda)$, for all $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$.

(ii) By (4.4.25),

$$\mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[V(s,\mu_s)\right] = V(t,\lambda) + \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^s \partial_t V(u,\mu_u) + \int_{\mathbb{R}^d} \mathbf{L}V(u,x,\mu_u,\alpha_u(x))\mu_u(dx)du\right]$$

By definition of $\hat{a}(t,x,\lambda)$, we have

$$\partial_t V(t,\lambda) + \int_{\mathbb{R}^d} \mathbf{L}V(t,\lambda,x,\hat{a}(t,x,\lambda)) + \psi(x,\lambda,\hat{a}(t,x,\lambda))\lambda(dx) = 0,$$

and so

$$\mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha^*}}\left[V(s,\mu_s)\right] = V(t,\lambda) - \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^s \int_{\mathbb{R}^d} \psi(x,\lambda,\alpha^*_u(x,\mu_u))\mu_u(dx)du\right].$$

By sending s to T , we then obtain

$$V(t,\lambda) = \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha^*}}\left[\Psi(\mu_T) + \int_t^T \int_{\mathbb{R}^d} \psi(x,\lambda,\alpha^*_u(x,\mu_u))\mu_u(dx)du\right] = J(t,\lambda,\alpha^*),$$

which shows that $V(t,\lambda) = J(t,\lambda,\alpha^*) \geq v(t,\lambda)$. Therefore, $V = v$ and α^* is an optimal Markovian control. □

4.4.5 Example of regular solution

For the following part of the paper, we suppose that there is no dependence on the measure for b, σ, and γ. With abuse of notation, we denote $b(x,a)$ (resp. $\sigma(x,a)$, $\gamma(x,a)$) instead of $b(x,\lambda,a)$ (resp. $\sigma(x,\lambda,a)$, $\gamma(x,\lambda,a)$). Fix $h \in C_b(\mathbb{R}^d)$ with $h(x) \geq 0$ for any $x \in \mathbb{R}^d$, and let $\Psi(\lambda) := \exp(-\langle h,\lambda\rangle)$ and $\psi = 0$. Therefore, the cost function J writes as

$$J(t,\lambda,\alpha) \quad = \quad \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\exp(-\langle h,\mu_T\rangle)\right], \tag{4.4.31}$$

for $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$.

The following assumption will ensure that there exists a smooth solution to the HJB equation (4.4.28) associated with this cost function.

Assumption A10. *Assume that the following conditions hold:*

(i) $h \in C^3_b(\mathbb{R}^d)$;

(ii) $\big(b,\sigma,\gamma\big)(\cdot,a) \in C^2(\mathbb{R}^d)$ *for any* $a \in A$, *and* b, σ, *and* γ *and their partial derivatives are bounded on* $\mathbb{R}^d \times A$*;*

(iii) there exists $C_\sigma > 0$ *such that*

$$\sigma\sigma^\top(x,a) \geq C_\sigma \boldsymbol{I}_d, \quad \textit{for } (x,a) \in \mathbb{R}^d \times A.$$

Proposition 4.4.29. *Under Assumption A10, there exists a function* $w \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$*, such that*

$$\begin{cases} -\partial_t w(t,x) - \sup_{a\in A}\left\{ b(x,a)^\top Dw(t,x) + \dfrac{1}{2}\mathit{Tr}\left(\sigma\sigma^\top(x,a)D^2 w(t,x)\right) - \dfrac{1}{2}\gamma(x,a)w(t,x)^2 \right\} = 0 \\ w(T,x) = h(x). \end{cases} \tag{4.4.32}$$

Moreover, we have

$$v(t,\lambda) = \exp\left(\langle w(t,\cdot),\lambda\rangle\right)$$

for any $(t,\lambda) \in [0,T]\times M\big(\mathbb{R}^d\big)$*, with* v *as in* (4.3.21)*.*

Proof. Our goal is to determine the function w using [106, Theorem 6.4.4], which guarantees the existence of smooth solutions for a certain category of fully nonlinear partial differential equations. First, we need to modify (4.4.32) to fall into this class. If w satisfies (4.4.32), we see that the function $\tilde{w}(t,x) := e^{-t}w(t,x)$ satisfies the following nonlinear PDE

$$\begin{cases} -\partial_t \tilde{w}(t,x) - \sup_{a\in A}\Big\{ b(x,a)^\top D\tilde{w}(t,x) + \dfrac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,a)D^2\tilde{w}(t,x)\right) \\ \qquad\qquad - \tfrac{1}{2}\gamma(x,a)e^t\tilde{w}(t,x)^2 + \tilde{w}(t,x)\Big\} = 0 \\ \tilde{w}(T,x) = e^{-T}h(x). \end{cases} \tag{4.4.33}$$

Let $C_\gamma > 0$ be a constant such that

$$\gamma(x,a) \geq C_\gamma, \quad \text{for all } (x,a) \in \mathbb{R}^d \times A.$$

Without loss of generalities, we can take C_γ to be such that $\frac{C_\gamma e^T}{4} > 1$. This means that

$$\begin{aligned} &\frac{1}{2}\gamma(x,a)e^t M_0^2 - M_0 \leq \frac{C_\gamma e^T}{2}M_0^2 - M_0 \leq -\delta_0, \\ &\frac{1}{2}\gamma(x,a)e^t M_0^2 + M_0 \geq M_0 \geq \delta_0, \end{aligned}$$

for all $(x,a) \in \mathbb{R}^d \times A$, with $M_0 := \frac{4}{C_\gamma e^T} > 0$ and $\delta_0 := M_0^2$. Combining these inequalities with Assumption A10, the property that equation (4.4.33) belongs to the class of [106, Theorem 6.4.4] follows from [106, Example 6.1.8]. Therefore, this theorem ensures that there exists $\tilde{w} \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$ solution to (4.4.33). Thus, $w(t,x) := e^t\tilde{w}(t,x)$ is a bounded solution of (4.4.32) belonging to $C_b^{1,2}([0,T]\times\mathbb{R}^d)$.

We conclude this proposition by applying Theorem 4.4.15. Define $V(t,\lambda) := \exp\left(\langle w(t,\cdot),\lambda\rangle\right)$. Using the terminal condition of w, we see that $V(t,\lambda) = \Psi(\lambda)$. Moreover, (4.4.28) in this setting writes as

$$\exp\left(\langle w(t,\cdot),\lambda\rangle\right)\int_{\mathbb{R}^d}\Bigg(-\partial_t w(t,x)-\sup_{a\in A}\Big\{b(x,a)^\top Dw(t,x)+\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,a)D^2w(t,x)\right)$$
$$-\frac{1}{2}\gamma(x,a)w(t,x)^2\Big\}\Bigg)\lambda(dx)=0.$$

It is now clear that V satisfies (4.4.28) since w satisfies (4.4.32). Moreover, the optimal control $\hat{a}$, defined as in (4.4.30), is the point that reaches the maximum over a compact set of a continuous function. This means that $\hat{a}$ can be chosen continuously, thus predictably. Therefore, its associated optimal control belongs to $\mathcal{A}$. We can now apply Theorem 4.4.15 and conclude. □

Remark 4.4.12. *When we are not under Assumption A10,* (4.4.32) *can be solved in a viscosity sense. In particular, one can follow [42, Section 5] to prove a comparison principle and an approximation procedure to give a solution in a viscosity sense to this equation. This translates directly to* (4.4.28) *and is a way to entail viscosity properties reducing the dimensionality.*

4.5 Conclusion

This research centers on controlled superprocesses, which is, to the best of our knowledge, a novel category of processes. The first part of our study is devoted to introducing the formalism, which we present in a weak form through a controlled martingale problem. Following the definition, we prove their existence and uniqueness in law. Uniqueness uses the same method as in Chapter 3, where the initial martingale problem is generalized to define càdlàg processes with values in $M(\mathbb{R}^d)$. Subsequently, we employ the duality method to establish that there exists at most one probability satisfying the martingale problem once the control and starting conditions are fixed. In the second part, we prove the existence of these processes as weak limits of rescaled branching processes. Thus, we introduce this new class of processes, whose existence has been proved in Chapter 3. We prove the weak limit using the Aldous criterion (see, *e.g.*, [54, 72, 141]) while adapting these ideas to this new setting. We notice that there are several choices for the branching parameters to build the class of superprocesses as limit points. We refer to [54, 72, 133, 141] for more details.

Once we establish the non-explosion property with respect to the chosen distance that metricizes weak* topology, we introduce the control problem. We define a weak formulation for controlled superprocesses as in Chapter 3. This allows us to extend the methods in [69, 70] to this setting and prove the DPP.

In the final section, we derived an HJB equation on the space of measures. Adopting the differential calculus developed in [118], we generalized the original martingale problem to a larger class of functions. This enabled us to derive the HJB equation and provide a verification theorem. Finally, we used this result to introduce a class of solvable problems. Utilizing the branching property technique, we translated the problem on finite measures to a finite-dimensional nonlinear PDE, reducing the dimensionality. We solved the latter using results from [106], and, with the help of the verification theorem, we provided an explicit description of the value function.

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Dynamic optimization of branching diffusion processes
Stochastic Control's lens on particle systems and their scaling limits

Abstract

The goal of this thesis is to uncover interesting structures occurring in the intersection of three distinct fields: stochastic control theory, branching diffusion processes, and McKean–Vlasov dynamics. In the initial phase, we investigate potential extensions of the stochastic target problem and the optimal stopping problem within the context of branching processes. By constraining our examination to cost functions that respect the inherent symmetry of the problem, we show how the optimization of a global criterion can be recast as finite-dimensional optimization challenges through the utilization of a branching property. This finding paves the way to a differential characterization. Using a dynamic programming approach, we prove the value function is the unique viscosity solution to an HJB equation.
The second part of this work delves into the theory of controlled branching diffusion processes, under a symmetrical structure in the cost function with respect to particle labeling. Exploring a relaxed formulation, we rewrite the control problem as the minimization of a lower semicontinuous function within a compact domain. This formulation, therefore, provides theoretical guarantees regarding the existence of a globally optimal solution. This abstract setting paves the way to scaling limits for these processes, leading to the class of controlled superprocesses. Within this dynamical framework, we establish an HJB equation in the space of finite measures. Moreover, for specific cost functions, we go back to the initial approach, retrieving regular solutions for the control problem through a branching property and finite-dimensional optimization.

Keywords: stochastic control, stochastic target control, optimal stopping, relaxed control, branching diffusion process, superprocesses, dynamic programming principle, Hamilton–Jacobi–Bellman equation, viscosity solution, martingale representation

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Résumé

Cette thèse se trouve à l'intersection de trois sujets différents : la théorie du contrôle stochastique, les processus de diffusion branchants et la dynamique de McKean–Vlasov. Initialement, nous étudions les extensions du problème de la cible stochastique et du problème de l'arrêt optimal pour des processus de branchement. Pour des fonctions de coût qui respectent la symétrie inhérente au problème, nous montrons comment l'optimisation d'un critère global peut être transformée en un problème à dimension finie grâce à l'utilisation d'une propriété de branchement. Cette constatation ouvre la voie à une caractérisation différentielle. En utilisant une approche de programmation dynamique, nous prouvons que la fonction de valeur est l'unique solution de viscosité d'une équation de HJB.
La deuxième partie de ce travail approfondit la théorie des processus branchants contrôlés, sous une structure symétrique de la fonction de coût par rapport à l'étiquette des particules. En explorant une formulation relâchée, nous réécrivons le problème de contrôle comme la minimisation d'une fonction semi-continue inférieurement à l'intérieur d'un compact. Ce point de vue fournit donc des garanties théoriques quant à l'existence d'une solution optimale. Ce cadre abstrait ouvre la voie à des limites d'échelle pour ces processus, conduisant à la classe des superprocessus contrôlés. Nous établissons ainsi une équation de HJB dans l'espace des mesures finies. De plus, pour des fonctions de coût de type exponentiel, nous revenons à l'approche initiale, retrouvant des solutions régulières pour le problème de contrôle grâce à une propriété de branchement et à une optimisation en dimension finie.

Mots clés : contrôle stochastique, cible stochastique, arrêt optimal, contrôle relaxé, processus de branchement de diffusion, superprocessus, principe de programmation dynamique, équation de Hamilton–Jacobi–Bellman, solution de viscosité, représentation martingale

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